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KEY

TEACHER'S HAND-BOOK OF ALGEBRA.

McLELLAN.

W. J. Gage & Co's Mathematical Series.

KEY

TO

THE TEACHER'S HAND-BOOK OF ALGEBRA

BY

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PREFACE.

At the request of many teachers who wished to use the Hand-Book as a practice-book for their pupils, no answers and but few hints were given in the book, but it has been thought desirable to prepare this "Companion" which will fully meet the wants of both Teacher and Private Student, as expressed in numerous suggestions from Teachers and Students throughout Ontario. Full solutions are given of all the difficult problems, and in every case, the hints and steps are such as to meet the requirements of the Student. It is believed that the Key will prove useful not only in lessening the labour of Teachers, but also in assisting that large class of Students who are endeavouring, without the aid of a Teacher, to obtain a thorough grounding in Elementary Algebra.

To J. Ryerson, M.A., of Barrie Collegiate Institute, J. C. Harstone, M.A., of Port Hope High School, and Mr. Jas. Miller, of Bowmanville High School, my thanks are due for valuable assistance in the preparation of the Key.

J. A. M.

CHAPTER I.

Exercise i., page 1.

1. 9; -69; 1; 0; 1206; -29; $1\frac{3}{5}$.
2. -160; 106; 41; 108.
3. $-\frac{5}{19}$, $\frac{13}{35}$, -25, 125; $\frac{7}{19}$, -31, $-4\frac{1}{81}$: 0, -1.
4. 9, 3; 7, $-\frac{1}{22}$.
5. 176; 82, $254\frac{26}{27}$; $-37 \div 7\sqrt[3]{3}$.
6. .18 each.
7. 146; 14; -72; -270; 396.
8. Each $=0$.

Exercise ii., page 7.

1. -1; 2. -166542; 3. 100.
4. -2967511; 5. 968; 6. -162 7. 10
8. -8; 9. 0; 10. -20; 11. 706440254900.

12.

	$1+3$	-13	-38	
$+3$	3	$+19.75329$	$+31.8043$	
$.5$	$\cdot 5$	$+\ 3.29222$	$+\ 5.3007$	
8	8	$.52675$	$.8481$	
4	4	2634	424	
4	4	263	42	
3	3	20	3	
	$1+6\cdot 58443$	$+10.6014$;	0	*Ans.*

-3	$1+3$	-13	-38	
$.7$	-3.77931	$+\ 2.3379$	$+30.165$	
7		$.5455$	7.038	
9		545	$.704$	
3		69	90	
1		2	3	
	$-\ .77931$	-10.055 ;	0	*Ans.*

−2	1+3	−13	−38
.8	−2.80512	− .3898	+27.0934
0		.1559	+10.8374
5		10	677
1			13
2			3
	1+ .19488	−13.5467;	0 *Ans.*

13. 0 for each.

Exercise iii., page 7.

1. $0,\ 16a^4$; 2. $a,\ a\sqrt{3}$; 3. $2a,\ 0$;

4. $26a^6,\ -26a^6$; 5. 0; 6. $4a^4$; 7. $6a^4$;

8. $\frac{3}{2}$; 9. c; 10. 0; 11. $\frac{a}{a+b}$;

12. $\frac{a^2c}{b^2}(b+2c)$; (13). $a^2+b^2+c^2$; 14. 0.

15. From the value of x, the *cube* in the given expression becomes $\left(\frac{a+b}{3b-a}\right)^3$; also $x+2a+b=\frac{3(a+b)}{2}$, and $x-a-2b=-\frac{3(a+b)}{2}$; $\therefore$ the second fraction in the given expression $=-1$, and the result is $\left(\frac{a+b}{3b-a}\right)^3+1=(12a^2b-24ab^2+28b^3)\div(3b-a)^3$.

16. 0; 17. 0; 18. $-b^2c$; 19. 20, 21, 22, each $=0$.

23. The square of the difference of two quantities is equal to the sum of their squares diminished by twice their product.

24. $(a+b)(a-b)=a^2-b^2$; 25. $2(l+b)h,\ 4x^2$.

26. The product of two binomials which have one common term, is equal to the square of the common term, *plus* the sum of the unlike terms multiplied by the common term, *plus* the product of the unlike terms.

27. $(a+b)^2-2ab=a^2+b^2$.

28. The sum of the cubes of two quantities is equal to the sum of the cubes of the quantities, increased by three times their product multiplied by their sum.

29. $(x-y)^3 = x^3 - y^3 - 3xy(x-y.)$ 30. $\dfrac{x^3+y^3}{x+y} = (x-y)^2 + xy.$

31. The difference of the cubes of two quantities divided by the difference of the quantities, is equal to the square of the sum of the quantities, diminished by their product.

32. $d^2 = 3l^2.$ 33. $l = \sqrt{(\frac{1}{3}d^2)}.$

34. Area $= \frac{1}{2}\sqrt{(2m^2)} \times \sqrt{(2l^2)}$, where l, m, are the sides of rect.

35. πr^2, $\pi(r^2 - r'^2) = \pi(r+r')(r-r')$, where r, r' are the radii.

36. $\pi r^2 h$, $\frac{1}{3}\pi r^2 h$, $\frac{2}{3}\pi r^2 h$.

Exercise iv., page 10.

1. $2(bx+cy).$ 2. $3(ax-by).$

3. $a^2(x-z) - ab(x-y) - b^2(y-z).$ 4. $(x+y+z)(a+b+c).$

5. $(a+b+c)(x^2+y^2+z^2).$

6. $2(x+y+x)(a^2+b^2+c^2-ab-ac-bc).$ 7. $0.$

8. $2(ax+cy+bz).$ 9. $a^2+b^2+c^2.$

10. $2x^n(a-2b).$ 11. $(a+b-c)$

Exercise v., page 12.

1. $2(x^2+9y^4)$, $4a^2b^2.$ 4. $4(a^2-b^2)^2.$

5. x^2+4x, $-3\frac{1}{4}x^4 - 4x^2y^2 + 3\frac{1}{4}y^4.$ 6. $a^2.$

8. $x^2 - 6x^3 + 9x^4 + 2xy - 6xy^2 - 6x^2y + 18x^2y^2 + y^2 - 6y^3 + 9y^4$

9. $4xy(x^2-y^2)$, $2(1+12x^2+16x^4.)$ 10. $\frac{1}{16}c^2.$

11. a^2-2b^2, $8ab(a+b)^2.$ 12. $2(a-c)(b-d).$

13. $\frac{1}{4}x^2 + \frac{1}{4}y^2 + \frac{1}{4}z^2 + \frac{1}{2}(xy+yz+zx)$ 15. $(1+x^2)^2.$

16 $4(xy+yz+zx) - 2(x^2+y^2+z^2).$ 17. $x^2.$

18. $(a^2+2b^2-2c^2)^2.$ 19. $16x^2y^2.$

20. $-4ab.$ 21. $4(a+b+c)^2.$

23. $4(1+x^2+x^4+x^6).$ 24. $(a^2x^2+b^2y^2)^2.$

Hints and Solutions.

1, 2, 3. See Ex. 1.

4. The quantity in first brackets is seen to be $(a+3b+a-b)^2 = 4(a+b)^2$, now multiply this by $(a-b)^2$.

5. Actual expansion, *or* use formula [4].

7. Actual expn. of given expression, *or* by symmetry.

9. See Ex. 1.

10. The expression is seen to be $(2a-b+2b-c+2c-a)^2 = (a+b+c)^2 = \&c.$

12. Actual expansion, *or* by formula [4]

$$(a+2b+c)(a-c)-(c+2d+a)(a-c) = \&c.$$

13. Expression is seen to be $(\frac{1}{2}x-y+\frac{1}{2}y-z+\frac{1}{2}z-x)^2 = \&c.$

14. Transpose left hand member, and expression becomes $(x-y+y-z+z-x)^2=0$, &c.

15. Expn. $=(1+x)^4-2(1+x^2)(1+x)^2+(1+x^2)^2+(1-x^2)^2=\{(1+x)^2-(1+x^2)\}^2+(1-x^2)^2$, &c. See Ex. 4.

17. $\{(x-2y+3z)-(3z-2y)\}^2 = \&c.$

18. Expn. $=\{(a^2+b^2-c^2)-(c^2-b^2)\}^2 = \&c.$

19. Expn. $=\{(x+y)^2-(x-y)^2\}^2 = \&c.$

21. Given expn. $=(3a-b+3b-c+3c-a)^2 = \&c.$

22. Expn. $=(2x^2-y^2+2y^2-z^2+2z^2-x^2)^2=(x^2+y^2+z^2)^2 = \&c.$

23. Expn. $=2\{(1+x^3)^2+(1-x^3)^2+(x+x^2)^2+(x-x^2)^2\} = \&c.$ See Ex. 2.

24. Expn. $=(ax+by)^4-2(a^2x^2+b^2y^2)(ax+by)^2$

$$+(a^2x^2+b^2y^2)^2+(a^2x^2-b^2y^2)^2$$
$$=\{(ax+by)^2-(a^2x^2+b^2y^2)\}^2+(a^2x^2-b^2y^2)^2$$
$$=4a^2b^2x^2y^2+(a^2x^2-b^2y^2)^2 = \&c.$$ See Ex. 4.

Exercise vi.

1. $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$;
   $1-2x+3x^2-4x^3+3x^4-2x^5+x^6$.

2. $1-4x+8x^2-14x^3+14x^4-8x^5+5x^6+6x^7+x^8$;
   $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$.

3. $4a^2+b^2+c^4+1-4ab-4ac^2-4a+2bc^2+2b+2c^2$;
$1+x^2+y^2+z^2-2x+2y+2z-2xy-2xz+2yz$;
$\frac{1}{4}x^2+\frac{1}{9}y^2+36z^2-\frac{1}{3}xy+6xz-4yz$.

4. $x^6-2x^5y+3x^4y^2-4x^3y^3+3x^2y^4-2xy^5+y^6$;
$a^2x^2+2abx^3+(2ac+b^2)x^4+2(ad+bc)x^5+(2bd+c^2)x^6$
$+2cdx^7+d^2x^8$.

5. By actual expansion, *or* left hand member is seen to contain $(ay-bx)^2$ $\therefore$ by symmetry, &c.

6. Expression $=$

$\{(ax+by+cz)+(bx+cy+az)\}\{(ax+by+cz)-(bx+cy+az)\}$
$=(ax+by+cz)^2-(bx+cy+az)^2$

7. Multiplying by 2, we have to show
$(a-b)^2+(b-c)^2+(a-c)^2-2(a-b)(a-c)+2(a-b)(b-c)$
$-2(a-c)(b-c)=0$,
i.e., $\{(a-b)+(b-c)-(a-c)\}^2=0$.

8. $3(a^2+b^2+e^2)-2(ab+bc+ca)$.

9. Expand the given expression, and result can be put in the required form. *Or*, expression is symmetrical with respect to b and a_1, which may therefore be interchanged, giving the required result.

10. Put $a-b=x$, $b-c=y$, and $\therefore$ $a-c=x+y$; then we have to show $(xy-xy-y^2-x^2-xy)^2=x^2y^2+y^2(x+y)^2+x^2(x+y)^2$, *i.e.*, $(x^2+xy+y^2)^2=$ &c., then see Ex. 4.

11. $4a^2+\frac{1}{4}b^2x^2+\frac{1}{16}c^2x^2+4d^2x^2-2abx-acx+8adx$
$+\frac{1}{4}bcx^2-2bdx^2-cdx^2$.

12. Put $x=b-c$, $y=c-a$, and $\therefore$ $x+y=b-a=-z$, since $x+y+z=0$. Then $x^4+y^4+z^4=(b-c)^4+(c-a)^4+(a-b)^4$ $=$, by Ex. 6, $2(a-b)^2(b-c)^2+2(b-c)^2(c-a)^2+2(c-a)^2(a-b)^2$ $=2(x^2y^2+y^2z^2+z^2x^2)$; but $(x^2-y^2)^2+$ &c., $=2(x^4+y^4+z^4)-2(x^2y^2+y^2z^2+z^2x^2)=x^4+y^4+z^4$, by substituting from the first result.

13. The given expression reduces to $2(a^2b^2+b^2c^2+c^2a^2)$ $+4abc(a+b+c)$, which by Ex. 2, gives the result.

Exercise vii., page 15.

1. $(a^2-b^2)^2$; 2. $\frac{1}{4}x^4+y^4$; 3. $a^4+3a^2b^2+4b^4$.

4. x^4-y^4; 5. x^2; 6. $16x^2$; 7. 0.

8. $\{2a^2+(3b^2-4c^2)\}\{2a^2-(3b^2-4c^2)\}$
$=4a^4-9b^4-16c^4+24b^2c^2$.

9. $\{b+(2a-3c)\}\{b-(2a-3c)\}=b^2-9c^2-4a^2+12ac$;
$\{-3c+(2a-b)\}\{-3c-(2a-b)\}=9c^2-4a^2-b^2+4ab$.

10. x^8-y^8. 11. $x^8+x^4y^4+y^8$.

12. Expression $=(a+b)^2-(ab+1)^2=a^2-a^2b^2+b^2-1$.

13. First two factors are $(b^2+c^2-a^2)(b^2+c^2+a^2)$
$=(b^2+c^2)^2-a^4=b^4+c^4+2b^2c^2-a^4$
$=2b^2c^2$ by given condition.

The second pair of factors gives $\{a^2+(b^2-c^2)\}\times\{a^2-(b^2-c^2)\}=a^4-(b^2-c^2)^2=a^4-(b^4+c^4-2b^2c^2)=2b^2c^2$ by the given condition; $\therefore$ result is $2b^2c^2\times 2b^2c^2=4b^4c^4$.

14. $x^4+y^4+\frac{7}{16}x^2y^2$;

15. $\{x^4+3x^2+1-(2x^3+2x)\}\times\{x^4+3x^2+1+(2x^3+2x)\}$
$=x^8+2x^6+3x^4+2x^2+1$.

16. $4a^4x^2-4a^4xy+a^4y^2-a^2x^4-2a^2x^3y+2ax^5+2ax^4y-x^6$.

17. Apply formula [4] and result is at once obtained.

18. Using formula [4] the *sum* of the quantities whose squares are given, $=a^2+b^2+c^2+ab+bc+ca+a^2+ab+ac-bc=(a+c)^2+(a+b)^2$. And the *difference* of the quantities $=a^2+b^2+c^2+ab$ $bc+ca-a^2-ab-ac+bc=(b+c)^2$; $\therefore$ product is $\{(a+c)^2+(a+b)^2\}\times(b+c)^2=(a+c)^2(b+c)^2+(a+b)^2(b+c)^2=$ &c.

19. By formula [4] expression $=2(ab+cd)+a^2+b^2-c^2-d^2$ multiplied by $2(ab+cd)-a^2-b^2+c^2+d^2$. The *former* factor $=(a+b)^2-(c-d)^2=(a+b+c-d)(a+b-c+d)$. The *latter* factor $=(c+d)^2-(a-b)^2=(c+d+a-b)(c+d-a+b)$, $\therefore$ &c.

20. By formula [2] the first factor is $(x-y+z)^2$, and the second is $(x-y-z)^2$; $\therefore$ their product is $\{(x-y+z)(x-y-z)\}^2$ $=(x^2+y^2-2xy-z^2)^2$.

21. $x^8 - y^8$. 22. $\frac{1}{3} - 6a^2 + 27a^4$.

23. Expression $= (m+p+n+q)\{m+p-(n+q)\}$
$= (m+p)^2 - (n+q)^2 = \&c.$

24. First pair of factors gives $(x^2+x-1)(x^2+x+1) = x^4+2x^3+x^2-1$. The second pair gives $(1+x^2-x)\{1-(x^2-x)\} = 1-x^4+2x^3-x^2$. The only term exactly the same in these two results is $2x^3$, we have $\therefore$
$(2x^3+x^4+x^2-1)\{2x^3-(x^4+x^2-1)\} = 2x^2+x^4+2x^6-x^8-1$.

25. Expression $= (x^4+x^2y^2+y^4)^2 - (x^2y^2)^2 =$ by formula [4]
$(x^4+x^2y^2+y^4+x^2y^2)(x^4+x^2y^2+y^4-x^2y^2)$
$= (x^2+y^2)^2(x^4+y^4) = x^4(x^2+y^2)^2 + y^4(x^2+y^2)^2$
$= (x^4+x^2y^2)^2 + (y^4+x^2y^2)^2$. See Ex. 4.

Exercise viii., page 16.

1. $x^4+4x^3+3x^2-2x-12$; $x^2+y^2-2xy+8xz-8yz+15z^2$.

2. $x^4+12x^3+49x^2+78x+40$; see Ex. 2; $x^6+bx^3-a^2+3ab-2b^2$.

3. $a^8+8a^6-10a^4-104a^2+105$; $x^8+2x^6-x^2-2$.

4. $x^4+5x^3y^2-12x^2y^2+5xy^3+y^4$.

5. $x^{2n}-2x^n-a^2-16a-63$; $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}+\dfrac{2x}{y}+\dfrac{2y}{x}-1$

6. $n^2x^2+2nxy+y^2+10nx+10y+21$.

7. $(x+a)^2+2y(x+a)-3y^2$.

8. $x^{4n}+2x^{3n}+x^{2n}(1-a-b)\quad x^n(a+b)+ab$.

9. $\frac{1}{4}x^8-x^4y^2+y^4-x^4+2y^2-8$.

10. $\left(\dfrac{1}{x}+\dfrac{1}{y}\right)^2+2\left(\dfrac{1}{x}+\dfrac{1}{y}\right)-\dfrac{5}{4}$.

11. First and third factors give $(x-2)^2-2 = x^2-4x+2$; the second and fourth give $(x-2)^2-3 = x^2-4x+1$, and $(x^2-4x+2)(x^2-4x+1) = (x^2-4x)^2+3(x^2-4x)+2$
$= x^4-8x^3+19x^2-12x+2$.

12. $(x+b+a)(x+b-a) = (x+b)^2-a^2$; and $(x+b-c)(x+b+c) = (x+b)^2-c^2$; product of these $= (x+b)^4-(a^2+c^2)(x+b)^2+a^2c^2$.

13. Expression $=(a+b)^2+(a+b)(c+d)+cd$
$$+(c+d)^2+(c+d)(a+b)+ab$$
$$-(a+b+c+d)^2=ab+cd.$$

14. Apply the formula. *Or* by symmetry as in Ex. 7, page 36. Type-terms are a^2 and ab; we see that there is no a^2 and $9ab$ $\therefore$ &c.

Exercise ix., page 17.

1. $2(1+3x^4)$; $2xy^3(3x^4+x^2y^6)$.

2. $9b(a^2+b^2+ab^2)$; $b(27a^2-27ab+7b^2)$ 3. $(x+y)^3$.

4. $8a^3$. 5. $8x^3$. 6. $8x^3$ 7. a^3. 8. $27x^3$. 9. $(2+x)^3$.

12. $8(x^2+y^2)^3$. 14. $(a^3+b^3)(x^3+y^3)$. 15. 0. 16. 0.

Hints and Solutions.

1 and 2. See example 1.

3. $=(x+y-z+z)^3$. See Ex. 2.

4. Sum of the two cubes is (Ex. 1.) $2a(a^2+3b^2)$, $\therefore$ &c.

5. Last term may be written $+3(x-y)(x+y)^2$ $\therefore$ expression $=(x-y+x+y)^3=(2x)^3=$ &c.

6. Expn. $=\{(1+x+x^2)-(1-x+x^2)\}^3=(2x)^3$. See Ex. 3.

7. Expn. $=(a-b-c+b+c)^3$. See Ex. 2.

8. Expn. $=\{(3x-4y+5z)-(5z-4y)\}^3=$ &c.

9. By formula [6] this is $\{(1+x+x^2)+(1-x)\}^3=$ &c.

10. First term of given expression is $a^4-6a^3b+12a^2b^2-8ab^3$; in this write a for b and b for a, and 2nd term is $b^4-6b^3a+12b^2a^2-8ba^3$, $\therefore$ difference is $a^4-b^4-6ab(a^2-b^2)+8ab(a^2-b^2)$ $=a^4-b^4+2ab(a^2-b^2)=(a^2-b^2)(a^2+b^2+2ab)$ $=(a-b)(a+b)^3$. *Or*, expand both sides of identity.

11. See last solution.

12. By formula [6] this $=\{(x^2+xy+y^2)+(x^2-xy+y^2)\}^3=$ &c.

13. Treating left-hand member as in 10 above, there results
$a^{12}+b^{12}+14a^3b^3(a^6+b^6)+51a^6b^6$, or
$a^{12}+b^{12}+2a^6b^6+14a^3b^3(a^6+b^6)+49a^6b^6$
$=(a^6+b^6)^2+14a^3b^3(a^6+b^6)+49a^6b^6$ which, formula [1]
$=(a^6+b^6+7a^3b^3)^2$.

14. Expand the first *cube* by formula [6], and last term of given expression cancels out, ∴ &c.

15. From the given condition $c=-(a+b)$; substituting this value of c in the given expression, it becomes $a^3+b^3-(a+b)^3+3ab(a+b)=(a+b)^3-(a+b)^3$, by formula [6].

16. As in last solution; $y^2=x^2+z^2$, substitute in given expression this value of y^2, &c.

Exercise x., page 19.

1. $1-3x+6x^2-7x^3+6x^4-3x^5+x^6$;
$a^3-b^3-c^3-3a^2(b+c)+3b^2(a-c)+3c^2(a-b)+6abc$;
$1-6x+21x^2-56x^3+111x^4-174x^5+219x^6-204x^7+144x^8-$
$-64x^9$.

2. $-(x^9+18x^3+27x^7+29x^6-24x^5-36x^4+5x^3-3x^2-2.)$

5. 0. 6. $45x^6+168\sum x^4y^2-432x^2y^2z^2$. 7. $(ax+by+cz)^3$.

Hints and Solutions.

1. Cube first two by [7]; last prob. may be treated as a binomial, and cubed by formula [6]; or see Ex. 4, p. 36.

3. Equating the right-hand members of formulas [8] and [9] the identity is at once derived.

4. See hint on Q. 14, Ex. 9

5. By formula [8] the first four terms of the given expression is seen to be $(x-2y+y-2z+z-2x)^3=-(x+y+z)^3$, hence, &c.

6. By actual expansion. But more easily by symmetry, as in Ex. 7, p. 36.

7. By formula [8] the given expression is seen to be

$$(2ax-by+2by-cz+2cz-ax)^3=\&c.$$

8. Cubing by formula [6], expression becomes $(x^3-y^3)^3+27x^6y^3+9x^2y\{(x^3-y^3)(x^3-y^3+3x^2y)\}+27x^3y^3\times(x+y)^3\}=(x^3-y^3)^3+9x^2y\{3x^4y^2+(x^3-y^3)^2+3x^2y(x^3-y^3)+3xy^2(x+y)^3\}$; now the quantity in the last brackets $=(x^3-y^3)^2+3xy(x^3y+x^4-xy^3+x^3y+3x^2y^2+3xy^3+y^4)$ $=(x^3-y^3)^2+3xy(x^2+xy+y^2)^2=(x^2+xy+y^2)^2\{(x-y)^2+3xy\}$ $=(x^2+xy+y^2)^3$; and $(x^3-y^3)^3=(x-y)^3(x^2+xy+y^2)^3$, $\therefore$ &c.

9. From formula [9] $(a+b+c)^3-3\sum a^2b=a^3+b^3+c^3+6abc$. Add $18abc$ to both sides, and this becomes

$$(a+b+c)^3-3\{a(b-c)^2+b(c-a)^2+c(a-b)^2\}$$
$$=a^3+b^3+c^3+24abc, \text{ i.e., } a^3+b^3+c^3-(a+b+c)^3$$
$$=-3\{a(b-c)^2+\ldots\ldots+\ldots\}-24abc\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From same formula, by transposing,

$$a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc$$
$$=\tfrac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}(a+b+c)+3abc.$$

Multiply both sides of this identity by 8, then $8(a^3+b^3+c^3)=4\{(a-b)^2+(b-c)^2+(c-a)^2\}(a+b+c)+24abc$.

Add this to (1) and the identity is found.

10. This is derived at once by putting $x+y+z=0$ in formula [9].

11. The condition is $x-2y-3z=0$; and result follows by writing in formula [9] $-2y$ for y, and $-3z$ for z.

12. This is same as last two: expression is divisible by $(x^2+xy+y^2)+(x^2-xy+y^2)+2z^2$, *i.e.*, by $2(x^2+y^2+z^2)$, &c.

13. In formula [8] write $a+b$ for x, $b+c$ for y, and $c+a$ for z, and the identity at once appears.

14. Cube by formula [6], and note that last term in right-hand member may be written $-3abxy(ax-by)$.

15. See Ex. 9 solved above.

16. Equate the two right-hand members of formulas [8] and [9], and in the result (see prob. 3 above) write $a+b-c$ for x, $b+c-a$ for y, and $c+a-b$ for z.

17. See solution of Ex. 10 above.

18. From 15 above (solved in 9) we have, by multiplying by 2 and subtracting $6abc$ from both sides,

$$2(a+b+c)^3-54abc=2(a^3+b^3+c^3-3abc)+6\{a(b-c)^2+\ldots+\ldots\}$$

Or substtuting for $2(a^3+b^3+c^3-3abc)$ its value from 15 above,

$$=\{(a-b)^2+(b-c)^2+(c-a)^2\}(a+b+c)+6\{a(b-c)^2+b(c-a)^2+c(a-b)^2\}=,$$ by addition, the required result.

19. In problem 3 of this exercise (derived as in 16) put $2a-b$ for x, $2b-c$ for y, and $2c-a$ for z, and the identity appears at once.

20. From the last *given* relation $2xyz=2abc$; add this to the three other given relations, then

$$x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz=2abc.$$

But by equating the right-hand members of formulas [7] and [8], we see at once that the *left*-hand member of the above

$$=(x+y)(y+z)(z+x).$$

NOTE.—The above identities can be proved as in Art. XXVI.

Exercise xi. page 21.

1. $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$;
$x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$,
$x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7$
$+y^8$.
$x^{12}+12x^{11}y+66x^{10}y^2+220x^9y^3+495x^8y^4+792x^7y^5+$
$924x^6y^6+792x^5y^7+$&c.

2. The signs will be alternately positive and negative.

3. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$; $a^4-8a^3b+24a^2b^2$
$-32ab^3+16b^4$; same as last, terms in inverse order.

4. $1+6m+15m^2+20m^3+15m^4+6m^5+m^6$; $m^5+5m^4+10m^3$
$+10m^2+5m+1$; $64m^6+192m^5+240m^4+160m^3+60m^2$
$+12m+1$.

5. 120. 6. $x^8-4x^6y+6x^4y^2-4x^2y^3+y^4$; $a^5-10a^4b^2+40a^3b^4-80a^2b^6+80ab^8-32b^{10}$; $a^{18}-12a^{15}b^3+60a^{12}b^6-160a^9b^9+240a^6b^{12}-192a^3b^{15}+64b^{18}$. 7. $495a^8b^4-792a^7b^5$.

8. Given expression $= 5x^4y+10x^3y^2+5x^2y^3+5xy^5$
$=5xy\{x^3+y^3+2xy(x+y)\}=5xy(x+y)(x^2-xy+y^2+2xy)$.

9. Multiply identity in 8 by 2, then

$$\begin{aligned}2\{(x+y)^5-x^5-y^5\}&=5xy(x+y)(2x^2+2y^2+2xy)\\&=5xy(x+y)\{x^2+y^2+(x+y)^2\}.\end{aligned}$$

In this identity put $a-c$ for x, $c-b$ for y, and *therefore* $a-b$ for $x+y$, then
$2\{(a-b)^5+(b-c)^5+(c-a)^5\}=5(a-b)(b-c)(c-a)\{(a-b)^2+\ldots+\ldots\}$. See examples page 13.

Exercise xii.

ANSWERS.

1. $1+x^3+x^4+x^6+x^{17}$.

2. $1+x+x^2+x^3+x^4+x^6+x^7+x^8+x^9+x^{15}$.

3. $x^4+2x^3-85x^2-86x+1680$;
$2x^9-3x^6+4x^5+x^4+x^3-2x^2-x+2$.

4. $x^6-57x^4+266x^2-1$.

5. $18x^8+21x^7+8x^6+x^5+63x^3+96x^2+43x+6$.

6. $1-\frac{1}{2}x^2-\frac{1}{8}x^4$.

7. $6x^{12}-4x^9-5x^8-2x^7+9x^6-10x^5+x^4-5x^3+5x^2+x+4$.

8. $x^3+9x^2+10x+11$. 9. x^4+3x^3. 10. x^4-3x^3.

11. x^4+8x^3-8x. 12. (1), -1; (2), -1; (3), -4. 13. -1.

Hints and Solutions.

Nos. 1—11 are intended as simple exercises in Horner's multiplication, but the student who has read the first three Chapters will notice short methods of solving several of them.

1. Omit x^3 from the second member

$$\begin{aligned}&(1+x+x^2+x^3+x^4)(1-x-x^7+x^8-x^{12}+x^{13})=\\&(1+x+x^2+x^3+x^4)(1-x)(1-x^7-x^{12})\\&=(1-x^5)\{1-x^7(1+x^5)\}=1-x^5-x^7(1-x^{10}).\end{aligned}$$

Now add the omitted $(1+x+x^2+x^3+x^4)x^3$.

[The above is the solution of the problem as it appears in the Hand-Book ; a misprint, however, occurred, the problem should have been $(1+x+x^2+x^3+x^4)(1-x^2+x^3-x^7+x^8-x^{12}+x^{13})$ of which the product is $1+x+x^3+x^4+x^{17}$.

2. $(1+x^5)(1+x+x^2+x^3+x^4)\{1-x^5(1-x)\}=$
$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9-x^5(1-x^{10})$.

1 (corrected) and 2 are particular cases of the general theorem $1+x+x^2+\ldots+x^{k-2}+x^{k-1}+x^{k+1}+x^{k+2}+\ldots+x^{m-1}+x^m+x^{mn+n+k}$ $=(1+x+x^2+\ldots+x^m)(1-x^k+x^{k+1}-x^{m+1+k}+x^{m+2+k}-x^{2m+2+k}+$ $x^{2m+3+k}-\ldots+x^{m(n-1)+n+k}$.

4. $\{(x^3-16x)+(5x^2-1)\}\{(x^3-16x)-(5x^2-1)\}=$
$(x^3-16x)^2-(5x^2-1)^2$.

6. $\sqrt{(1+x)}=1+\frac{1}{2}x-\frac{1}{8}x^2+\&c.$
$\therefore\ \sqrt{(1-x)}=1-\frac{1}{2}x-\frac{1}{8}x^2-\&c.,$ (by writing $-x$ for x),
and $\sqrt{(1-x^2)}=1-\frac{1}{2}x^2-\frac{1}{8}x^4-\&c.,$ (by writing $-x^2$ for x).

12. Arranging the terms of the sum in order

$$x^{16}+x^{15}+x^{14}+\ldots\ldots\ldots+x^2+x=-1,$$

or, $x^{16}+x^{15}+x^{14}+\ldots\ldots\ldots+x^2+x+1=0.$ (a)

Multiply both members by x and add 1 to each product.

$$x^{17}+x^{16}+x^{15}+\ldots\ldots\ldots+x^3+x^2+x+1=1 \qquad (b)$$

$(b)-(a)\ \therefore\ x^{17}=1$

$$\therefore\ x^{18}=x,\ x^{19}=x^2,\ x^{20}=x^3,\ \&c. \qquad (c)$$

The product of the first pair of factors is

$x^3+x^6+x^9+x^{10}+x^{12}+x^{13}+x^{15}+x^{16}+x^{18}+x^{19}+x^{21}+x^{22}$ $+x^{24}+x^{25}+x^{28}+x^{31}=\{\text{by }(c)\}x^3+x^6+x^9+x^{10}+x^{12}+x^{13}+$ $+x^{15}+x^{16}+x+x^2+x^4+x^5+x^7+x^8+x^{11}+x^{14}=-1.$

Similarly the other products may be found. Instead of writing the two lines as above, the student will find it convenient to write only the latter line ; this may be done by subtracting 17 at once from any index that rises above that number. Detached coefficients should also be used.

13. Arranging the terms of the sum,

$$x^{12}+x^{11}+x^{10}+\ldots\ldots\ldots+x^2+x=-1,$$

or $x^{12}+x^{11}+x^{10}+\ldots\ldots\ldots+x^2+x+1=0.$

Multiply both members by x and add 1 to each product.

$$x^{13}+x^{12}+x^{11}+\ldots\ldots\ldots+x^3+x^2+x+1=1,$$

$$\therefore\ x^{13}=1,\ x^{14}=x,\ x^{15}=x^2,\ x^{16}=x^3,\ \&c.$$

Multiply the given factors, subtracting 13 from any index greater than that number, the product is

$$5(x+x^2+x^3+\ldots\ldots\ldots+x^{11}+x^{12})+4x^{13}=-5+4=-1.$$

The following are other examples of this class of problems. In each case the sum of the factors is assumed to be equal to -1.

1. $(x+x^4)(x^2+x^3)=-1.$
2. $(x+x^6)(x^2+x^5)(x^3+x^4)=1.$
3. $(x+x^2+x^4)(x^3+x^5+x^6)=2.$
4. $(x+x^3+x^4+x^5+x^9)(x^2+x^6+x^7+x^8+x^{10})=3.$
5. $(x+x^3+x^4+x^9+x^{10}+x^{12})(x^2+x^5+x^6+x^7+x^8+x^{11})=-3.$
6. $(x+x^7+x^8+x^{11}+x^{12}+x^{18})(x^2+x^3+x^5+x^{14}+x^{16}+x^{17})(x^4+x^6+x^9+x^{10}+x^{13}+x^{15})=7.$
7. $(x+x^4+x^5+x^6+x^7+x^9+x^{11}+x^{16}+x^{17})$
$(x^2+x^3+x^8+x^{10}+x^{12}+x^{13}+x^{14}+x^{15}+x^{18})=5.$

Exercise xiii.

1. $3x^3-2x^2-4x+2.$
2. $5x^4-4x^3+3x^2-2x+1.$
3. $a^4+2a^3+3a^2+4a+5.$
4. $x^3+2x^2y+3xy^2+4y^3.$
5. $a^3+3a^2x+3ax^2+x^3.$
6. $4x^2+8x+7$; $-13x-20.$
7. $10x^3+5x^2+1$; $10x+10.$
8. $x^2-xy+y^2.$
9. $x^2-a^2.$
10. $x^4+(1-a)x^3+(1-a+b)x^2+(1-a)x+1.$
11. $3x^3+2x^2+x+1\frac{1}{2}$; $3\frac{1}{2}(x+1).$
12. $5x^2+13xy+12y^2.$
13. $6x^5-x^4-x^3+x^2-x+6$; $-1.$
14. $2x^4-3x^3+4x^2-5x+6.$
15. $a+b.$
16. $x+y+z.$
17. $10x^3$; $10(x^4-20).$
18. $mx^3+nx^2+a.$
19. $1+x-5\frac{3}{4}x^2-3x^3+9x^4.$
20. $33.$
21. $-4.$
22. $-20.$
23. $15y^4.$
24. $85x+8.$
25. $755.$

Hints and Solutions.

The whole of this exercise should be worked by Horner's methods.

1. Work like example 5, page, 27, also as follows: Multiply both divisor and dividend by -1 and arrange in ascending powers of x.

$$\begin{array}{r|rrrrrr} & 2 & -10 & 6 & 17 & -5 & -6 \\ 3 & & 6 & -12 & -6 & 9 & \\ 2 & & & 4 & -8 & -4 & 6 \\ \hline & 2 & -4 & -2 & 3. & & \end{array}$$

2. $\{(5x^5(x+1)+x^5+1\} \div (x+1)^2 = (5x^5+x^4-x^3+x^2-x+1) \div (x+1)$.

3 is merely a variation of 2.

5. $(a^2-x^2)^3 \div (a-x)^3 = (a+x)^3$.

7.
$$\begin{array}{r|rrrrrr} & 10 & 5 & -90 & -44 & 10 & 1 \\ 9 & & & 90 & 45 & 0 & 9 \\ \hline & 10 & 5 & 0 & 1; & 10 & 10. \end{array}$$

8. $\dfrac{x^6-y^6}{x+y} \div (x^3-y^3) = \dfrac{x^3+y^3}{x+y} = x^2-xy+y^2$.

9. $(x-a)^4(x+a)^2 \div (x+a)(x-a)^3 = (x-a)(x+a) = x^2-a^2$.

10. $\{(x^5-1)-ax(x^3-1)+bx^2(x-1)\} \div (x-1)$
$= (x^4+x^3+x^2+x+1)-ax(x^2+x+1)+bx^2$.

11.
$$\begin{array}{r|rrrrrr} & 6 & 7 & 7 & 6 & 6 & 5 \\ -1 & & -3 & -2 & -1 & -1\frac{1}{2} & \\ -1 & & & -3 & -2 & -1 & -1\frac{1}{2} \\ \hline 2 & 3 & 2 & 1 & 1\frac{1}{2}; & 3\frac{1}{2} & 3\frac{1}{2}. \end{array}$$

12.
$$\begin{array}{r|rrrrr} & 60 & 91 & 0 & -91 & 60 \\ 13 & & 65 & 169 & 156 & \\ -5 & & & -25 & -65 & -60 \\ \hline 12 & 5 & 13 & 12. & & \end{array}$$

This is a particular case of $\{(m^2-n^2)x^2+(m^2+n^2)xy+2mny^2\}$ $\{2mnx^2-(m^2+n^2)xy+(m^2-n^2)y^2\} = 2mn(m^2-n^2)(x^4+y^4)+$ $(n^2+2mn-m^2)(m^2+n^2)xy(x^2-y^2)$, a formula which the student may verify. In the case in the Hand-Book, $m=3$, $n=2$.

14.
$$\begin{array}{r|rrrrrrr}
 & 6 & -1 & 2 & -2 & 2 & 19 & 6 \\
-4 & & -8 & 12 & -16 & 20 & -24 & \\
-1 & & & -2 & 3 & -4 & 5 & -6 \\
\hline
3 & 2 & -3 & 4 & -5 & 6 & &
\end{array}$$

Work also in ascending powers of x.

15.
$$\begin{array}{l}
a^4+6a^3b+12a^2b^2+8ab^3 \\
\quad -8a^3b-12a^2b^2-6ab^3-b^4
\end{array}$$
$$\begin{array}{r|rrrrr}
\hline
 & 1 & -2 & 0 & 2 & -1 \\
3 & & 3 & 3 & & \\
-3 & & & -3 & -3 & \\
1 & & & & 1 & 1 \\
\hline
 & 1 & 1 & & &
\end{array}$$

16. $\{(x+y)+z\}^3 \div \{(x+y)+z\}^2$.

18. $\{m(bx^4+cx^3)+n(bx^3+cx^2)+a(bx+c)\} \div (bx+c) = mx^3+nx^2+a$.

19. Factor the divisor first.

$$\tfrac{1}{2}(2+11x-6x^2)=\tfrac{1}{2}(2-x)(1+6x).$$
$$\tfrac{1}{2}(2+13x-36x^3)=\tfrac{1}{2}(1+6x)(2+x-6x^2).$$
$$\tfrac{1}{4}(4-13x^2+6x^3)=\tfrac{1}{4}(2-x)(2+x-6x^2).$$
$$\therefore \text{ quotient}=\tfrac{1}{4}(2+x-6x^2)^2.$$

23. $\{(3x)^4-(2y)^4+15y^4\} \div \{(3x)-(2y)\}$.

Exercise xiv., page 31.

All these are done as in Exs, 1 and 2, Art. VIII.

1. y^3-2y^2-4y-9, if $y=x-3$.

2. y^3+3y+5, if $y=x+1$.

3. y^4+81, if $y=x-2$.

4. $y^4+4y^3-43y^2+92y-67$, if $y=x+2$.

5. $3y^5+30y^4+119y^3+238y^2+249y+106$, if $y=x-2$.

6. $y^4-\frac{59}{8}y^2-\frac{91}{8}y+\frac{845}{256}$, if $y=x-1\frac{3}{4}$.

7. $y^3 - \frac{16}{3}y + \frac{155}{27}$, if $y = x - \frac{2}{3}$.

8. $(x-2y^3) - 3y(x-2y)^2 - 18y^2(x-2y) - 24y^3$.

9. $(x-y)^5 - 10y^2(x-y)^3 - 20y^3(x-y)^2 - 10y^4(x-y)$.

10. $(2x+y)^3 + 2y^2(2x+y) + 5y^3$.

11. $512y^3 - 3y - \frac{19}{144}$, if $y = \frac{1}{8}x - \frac{1}{16}$.

12. $y^4 - 24y^2 + 49y - 28$, if $y = x+2$.

Sol. of 9.	1	−5	0	0	5	−1
1		+1	−4	−4	−4	+1
	1	−4	−4	−4	+1;	0
1		+1	−3	−7	−11	
	1	−3	−7	−11;	10	
1		+1	−2	−9		
	1	−2	−9;	−20		
1		+1	−1			
	1	−1;	−10			
1		+1				
	1;	0.				

CHAPTER II.

Exercise xv., page 33.

1. $a^2b+ab^2+a^2c+ac^2+b^2c+bc^2$; $(a-b)^2+(b-c)^2+(c-a)^2$; $a(b-c)+b(c-a)+c(a-b)$; $ab(x-c)+bc(x-a)+ac(x-b)$; $\Sigma a^3b^2c=\Sigma abc(a^2b)=abc(a^2b+a^2c+b^2c+ab^2+ac^2+bc^2)$; $(a+b)(c-a)(c-b)+(b+c)(a-b)(a-c)+(c+a)(b-c)(b-a)$; $\{(a+c)^2-b^2\}+\{(b+a)^2-c^2\}+\{(c+b)^2-a^2\}$; $a(b+c)^2+b(c+a)^2+c(a+b)^2$.

2. $abc+bcd+cda+dab$; $a^2(b+c+d)+b^2(c+d+a)+c^2(d+a+b)+d^2(a+b+c)$; $(a-b)+(b-c)+(c-d)+(a-d)+(a-c)+(b-d)$; $a^2(a-b)+b^2(b-c)+c^2(c-d)+d^2(d-a)$.

3. Write a for b and b for a, and there results $(x+b)(b+a)(a+x)+bax$, which is identical with given expn.

4. Interchange a and b, then $(b+a)^2+(b-a)^2$, which is identical with the given expn., it being remembered that $(b-a)^2=(a-b)^2$; interchange a and $-b$, then $(a-b)^2+(-b-a)^2=(a+b)^2+(a-b)^2$, $\therefore$ &c.

5. This is at once evident.

6. Change a into b, b into c, and c into a, then $b^2(c-a)-c^2(b-a)-a^2(c-b)$, which may be put in the form $a^2(b-c)-b^2(a-c)-c^2(b-a)$, the given expn.

7. Given expn. $=(a^2+b^2)(c^2+d^2)$, and the symmetry is evident.

8. Interchange x and y, the expression plainly is unaltered.

9. Interchanging x and y there results

$$\{y^3-x^3+3yx(2y+x)\}^3+\{x^3-y^3+3yx(2x+y)\}^3,$$

which is identical with given expn.

10. Interchange a and b, then $b(b+2a)^3+a(a+2b)^3$, which is identical; interchange a and $-b$, then

$$-b(-b-2a)^3-a(-a-2b)^3=b(b+2a)^3+a(a+2b)^3,$$

which is identical with the given expn.

11. Observe that the quantity within the *square* brackets is the *difference of two squares*, and by formula [4];

The *first* factor is

$$(a+c)(b+c)+2c(a+b)+(a-c)(b-c)=2(a+c)(b+c).$$

The second factor is

$$(a+c)(b+c)+2c(a+b)-(a-c)(b-c)=4c(a+b)$$

$\therefore$ the whole expn. $=ab\times 2(a+c)(b+c)\times 4c(a+b)$
$=8abc(a+b)(b+c)(c+a)$,

which is plainly symmetrical with respect to a, b, c.

12. In the given expression
$2abc(a+b+c)=2(a^2bc+b^2ac+c^2ab)=2(ab.ca+ab.bc+ca.bc)$.
Now interchanging ab, bc, ca, there results

$$b^2c^2+c^2a^2+a^2b^2+2(bc.ab+bc.ca+ab.ac)=\&c.$$

13. x and y. 14. ax and by; x, y, and z.

15. f and h. 16. x and y, also x and $-z$, and y and $-z$.

17. With respect to a, b, and $-c$; *i.e.*, a may be changed into b, b into $-c$, and $-c$ into a.

18. Symmetrical with respect x^2, $-y^2$, and z^2. *i.e.*, we may change x^2 into $-y^2$, $-y^2$ into z^2, and z^2 into x^2; this gives
$x^6+z^6-y^6-3(x^2+z^2)(-z^2+y^2)(-y^2+x^2)=$
$z^6-y^6+x^6-3(x^2+z^2)(y^2-z^2)(y^2-x^2)$,
which is identical with given expression. *Or*, by [8], expn. $=(x^2-y^2+z^2)^3$.

19. b and c. 20. a and c. 21. a and b.

22 a^2, $2ab$. 23. a^2b, abc. 24. a^2b, abc.

25 x^5, x^4y, x^3y^2. Same. x^4y, x^3y^2;

26. Not symmetrical, $\therefore$ none. Same. 27. x^4, x^3y, x^2y^2, x^2yz.

28. a^4, a^3b, a^2bc, $abcd$. a^4, a^2b^2. 29. a^3, a^2b.

Exercise xvi., page 37.

1. $4(a^2+b^2+c^2)$.
2. $3(a^2+b^2+c^2)+2(ab+bc+ca)$.
3. $4(a^2+b^2+c^2+d^2)$.
4. $2(a^2+b^2+c^2)$.
5. $4(x^2+y^2+z^2+n^2)$.
6. $2(a^3+b^3+c^3)+6\Sigma a^2b-12abc$.
7. $14(x^2+y^2+z^2)+2(xy+yz+zx)$.
8. $24abcmnr$.
9, $2abc(a+b+c)$.
10. $a^2b^2+b^2c^2+c^2a^2$.

Hints and Solutions.

1. Type-terms a^2, ab; former occurs in each of the four terms; $2ab$ occurs in first and second, $-2ab$ in third and fourth, $\therefore$ &c.

2. Type-terms as last, a^2 in each term; $-2ab$ in first and second terms, $+2ab$ in last, result $-2ab$, $\therefore$ &c.

3. Type-terms a^2, ab; a^2 in each term, $2ab$ in first and third, $-2ab$ in second and fourth, $\therefore$ term ab vanishes, &c.

4. Type-terms a^2, ab; a^2 in first and second terms, $2ab$ in first, $-ab$ in second and third, $\therefore$ ab vanishes, &c.

5. Type-terms x^2, xy; x^2 in each term, $2xy$ in first and last, $-2xy$ in second and fourth, $\therefore$ xy vanishes, &c.

6. Type-terms a^3, a^2b, abc; a^3 in first, second and fourth, $-a^3$ in third, $\therefore$ $2a^3$; $3a^2b$ in first three terms, $-3a^2b$ in last, $\therefore$ $6a^2b$; $6abc$ in first, and $-6abc$ in each of the other three, $\therefore$ $-12abc$, &c.

7. Symmetrical in x, y, z; x^2, xy type-terms; x^2 in first term, $9x^2$ in second, and $4x^2$ in third, $\therefore$ $14x^2$; $-4xy$ in first, $-6xy$ in second, and $+12xy$ in third, $\therefore$ result is $2xy$, &c.

8. Symmetrical in ma, nb, rc; type-terms m^3a^3, m^2a^2nb, $abcmnr$; m^3a^3 in first and last, and $-m^3a^3$ in second and fourth, $\therefore$ this term vanishes; $3m^2a^2nb$ in first and last, and $-3m^2a^2nb$ in second and third, $\therefore$ this term vanishes; $6abcmnr$ in each of the four terms, $\therefore$ &c.

9. Type-terms are a^2bc, ab^3; ab^3 in first, $-ab^3$ in second, $\therefore$ this term vanishes; a^2bc in first and second, $\therefore$ $2a^2bc$, &c.

10. Type-terms are a^2b^2, a^2bc; $2a^2bc$ in first term, $-2a^2bc$ in second, $\therefore$ this vanishes; a^2b^2 in first term, $\therefore$ &c.

11. Type-terms are a^2x^2, $abxy$; a^2x^2 occurs twice, and $2abxy$ occurs twice, $\therefore$ $2(a^2+b^2+c^2)x^2$, and $4(ab+bc+ca)xy$, and the same for y^2, z^2, and yz, zx, $\therefore$ &c.

12. Type-terms are a^4, a^3b, a^2b^2, a^2bc; these are obtained by taking terms of the expansion; thus

$$(a+b+c)^4=(a+b)^4+4(a+b)^3c+6(a+b)^2c^2+\ldots;$$

a^4 in each term; $4a^3b$ in first and last terms, $-4a^3b$ in second and fourth, $\therefore$ this term vanishes; $6a^2b^2$ in each of the terms $=$ $24a^2b^2$; $12a^2bc$ occurs in first two terms, and $-12a^2bc$ in last two, $\therefore$ this term vanishes; hence result.

13. $(a+b+c)^4=(a+b)^4+4(a+b)^3c+6(a+b)^2c^2+$ &c.

$$=a^4+4a^3b+6a^2b^2+\ldots\ldots+4c(a^3+3a^2b+\ldots)+\ldots$$

Hence the type-terms are a^4, $4a^3b$, $6a^2b^2$, $12abc$, &c.

14. $(\Sigma a)^4=(a+b+c+d+\ldots)^4=\{(a+b+c)+d+\ldots\}^4$

$$=(a+b+c)^4+4(a+b+c)^3(d+\ldots)+\ldots$$

The first of these powers gives the type-terms found in last Ex., and the second gives the only other possible type-term, $abcd$, with coefficient 24; $\therefore$ &c.

15. $(a^2+b^2+c^2)^3=\Sigma a^6+3\Sigma a^4b^2+6a^2b^2c^2.$

$2(ab+bc+ca)^3=2\Sigma a^3b^3+6\Sigma a^3b^2c+12a^2b^2c^2.$

$-3(a^2+b^2+c^2)(ab+bc+ca)^2=-3\Sigma a^4b^2-6\Sigma a^3b^2c-9a^2b^2c^2-6\Sigma a^4bc.$

Now add, and the given expression

$$=\Sigma a^6+2\Sigma a^3b^3-6\Sigma a^4bc+9a^2b^2c^2=(\Sigma a^3-3abc)^2$$

$$=(a^3+b^3+c^3-3abc)^2.$$

A shorter solution is as follows:

By formula [9] we prove at once

$$a^3+b^3+c^3-3abc=\tfrac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}(a+b+c).$$

(See Exercise X., q. 15). Now the given expn. is same form as left-hand member of this expn.; for it is

$(a^2+b^2+c^2)^3+(ab+bc+ca)^3+(ab+bc+ca)^3-3(a^2+b^2+c^2)$ $(ab+bc+ca)^2=\tfrac{1}{2}\{(a^2+b^2+c^2-ab-bc-ca)^2+0+$

$(ab+bc+ca-a^2-b^2-c^2)^2\} \times (a+b+c)^2 =$
$(a^2+b^2+c^2-ab-bc-ca)^2(a+b+c)^2 = (a^3+b^3+c^3-3abc)^2$.

In the same way, Ex. 9, page 37, may be neatly done.

16. $\{(a-b)(b-c)\}^2 = (ab-b^2-ca+bc)^2$, and similar expns. for the other two terms; then proceeding as in other cases in the exercise, the result is $\Sigma a^4+3\Sigma a^2b^2-2\Sigma ab^3=(\Sigma a^2-\Sigma ab)^2$. Or, add to given expn., $2(a-b)(b-c)(c-a)(a-b+b-c+c-a)$ [which $=0$] and the result is the square required. (See Ex. 2, page 13. Also see Ex. 6, p. 18.)

17. This may be deduced from last by putting $a-b$ for a, $b-c$ for b, and $(c-a)$ for c.

18. Terms in $a^2r^2x^2$, $c^2s^2y^2$ vanish; also terms in

$$\left.\begin{matrix} abrsx^2, & bcrsy^2 \\ abxsr^2, & bcxys^2 \end{matrix}\right\} \text{vanish, and there remains}$$

$$acs^2x^2+acr^2y^2-2acrsxy+4b^2rsxy$$
$$-b^2(ry+sx)^2 = ac(sx-ry)^2-4b^2rsxy$$
$$-b^2(ry+sx)^2 = ac(sx-ry)^2-b^2(sx-ry)^2 = \&c.$$

19. a and b can be interchanged, and also c and d in left-hand member; we have a^2c^2, $\therefore$ also b^2c^2, and $\therefore$ a^2d^2 and b^2d^2, &c.; then this is shown = right-hand member.

20. In the left-hand side a may be written for b and product remains of the same form; also interchanging a and b and also c and d, the product will remain the same if a *minus* sign be placed before the second term.

21. Left-hand member is $6\Sigma w^4+12\Sigma w^2x^2$.

Type-terms in right-hand member are w^4, $4w^3x$, w^2x^2; w^4 is found in 6 terms $=6w$; $4w^3x$ in first, and $-4w^3x$ in second, $\therefore$ this term vanishes; $6w^2x^2$ in second, $\therefore$ this term vanishes; $6w^2x^2$ in first and $6w^2x^2$ in second $=12w^2x^2$, $\therefore$ right-hand member $=6\Sigma w^4+12\Sigma w^2x^2=$ left-hand member.

22. Type-terms in left-hand member are a^5, a^4b, a^3b^2, a^3bc, a^2b^2c; a^5 in first and second, $-a^5$ in third and fourth, $=0$; $5a^4b$ first and third, $-5a^4b$ in second and fourth $=0$; $10a^3b^2$ in

first and second, $-10a^3b^2$ in third and fourth $=0$; $2a^3bc$ in all the terms. $\therefore$ result $=\frac{1}{5}(80\Sigma a^3bc)=16abc(a^2+b^2+c^2)$. Similarly first factor of right-hand member may be proved $=8abc$, and the second $=2(a^2+b^2+c^2)$, $\therefore$ &c. The problem may be more easily solved as in Ex. 4, p. 43.

Exercise xvii., page 40.

1. 115. 2. $pa^3-3qa^2+3ra-s$. 3. 2. 4. $-17{\cdot}3538$.
5. 1, $2(3a^2+1)$. 6. 0 or $2y^n$, $2y^n$, 0. 7. 36.
8. $-(b^2+a^2)^3-(3b^2)^3$. 9. $-15a^4$. 10. $3888a^4b^4$.
11. $a^2b^2(a+b)$. 12. 0.
13. $2a^3-3ab(a-b)$; $2b^3+6ab(a+b)$; $2(a^3+b^3)$

Hints and Solutions.

1. Find remainder by Horner's division.
2. Write a for x in given expression.
3. Find remainder by Horner's division.
4.
$$\begin{array}{r|rrrrrr} & 10 & -20 & -10 & -{\cdot}89 & -8{\cdot}9 & +20 \\ 11 & & 11 & 9.9 & 21{\cdot}89 & -25{\cdot}058 & -37{\cdot}3538 \\ \hline 10 & 1 & -{\cdot}9 & -\ 1{\cdot}99 & -2{\cdot}278 & -\ 3{\cdot}3958; & -17{\cdot}3538 \end{array}$$

Observe that the zero coefficients in the dividend need not be inserted if the zero coefficients of divisor be not considered.

5. $(-1+1)^5-(-1)^5=1$; writing -2 for x, we have $(a+1)^3-(a-1)^3=$ &c.

6. $(-y)^n+y^n=0$ or $2y^n$ as n is odd or even; $(-y)^{2n}+y^{2n}=2y^{2n}$; $(-y)^{2n+1}+y^{2n+1}=0$.

8. Expn. $=(x^2-a^2)^3+(x^2-2b^2)^3$, in which write $-b^2$ for x^2.

9. In the first pair of factors write $-2a^2$ for x^2 and we get $(ax-a^2)(-ax-a^2)=-a^2x^2+a^4=3a^4$ (by substituting for x^2).

In the second pair of factors x^2+2a^2 may be at once struck out, giving $-(-3ax)\times 3ax=9a^2x^2=-18a^4$ (by substituting for x^2).

10. Divisor $=0$ gives $9a^2+4b^2=12ab$, whence $81a^4+16b^4=72a^2b^2$; substituting these values in given expression we have $18ab \times 6ab \times 36a^2b^2 = 3888a^4b^4 = 768b^8$. But observe that this is the remainder on dividing by $3a-2b$. To get the true remainder on dividing by $9a^2-12ab+4b^2$, we must divide the given expression $(=6561a^8+1296a^4b^4+256b^8)$, we may use Horner's division, thus:

$$\begin{array}{r|rrrrrrrrr}
 & 6561 & 0 & 0 & 0 & 1296 & 0 & 0 & 0 & 256 \\
+12 & & & & & & & & & \\
-\ 4 & & & & & & & & & \\
\hline
9 & 729, & 972, & 972, & 864, & 864, & 768, & 640; & 4608 & -2304
\end{array}$$

where the coefficients of quotient are all positive, and the remainder is $4608ab^7-2304b^8$. If in this remainder we substitute the value of $3a$ $(=2b)$ we get $768b^8$, the rem. on dividing by $3a-2b$.

11. Put $a+b$ for x; $\therefore a^2(a+b-a)^3+b^2(a+b-b)^3=$&c.

12. Given expression (cubing by formula [6])
$=(a^3+b^3)(x^3+y^3)$, of which one factor is exactly divisible by $a+b$, and the other by $x+y$.

13. Put $a-b$ for x, then $(a-b)^3+a^3+b^3-3ab(a-b)$
$=a^3-b^3-3ab(a-b)+a^3+b^3-3ab(a-b)$
$=2a^3-3ab(a-b)$; in second case put $-(a-b)$ for x,
$\therefore\ -(a-b)^3+a^3+b^3+3ab(a-b)=2b^3+6ab(a-b)$.
Or, interchange a and b in the first result; in the third case put $a+b$ for x, $\therefore (a+b)^3+a^3+b^3-3ab(a+b)=2(a^3+b^3)$.

14. Let the polynome be $ax^n+bx^{n-1}+\ldots\ldots+hx+k$; to find remainder put 1 for x, result is $a+b+\ldots\ldots+h+k$.

Exercise xviii., page 41.

1. Dividend $=x^{2n+1}-(-y)^{2n+1}$ which is divisible by $x-(-y)$; dividend $=x^{2n}-(-y)^{2n}$, and divisor $=x-(-y)$, $\therefore$ by the Cor., &c.

2. Dividend $=(x^4)^3-(-y^4)^3$, and divisor $=x^4-(-y)^4$;
dividend $=(x^6)^5-(-y^6)^5$, and divisor $=x^6-(-y)^6$;
dividend $=(x^{10})^3-(-y^{10})^3$, divisor $=x^{10}-(-y)^{10}$;
dividend $=(x^2)^{15}-(-y^2)^{15}$, and divisor $=x^2-(-y^2)$.

3. Dividend $=(ax+by)^5-(-bx-ay)^5$, divisor
$=ax+by-(-bx-ay)$. $\therefore$ by Cor., &c.

4. Divisor $=ax+by+cz-(bx+cy+az)$, $\therefore$ &c.

5. Expression is divisible by $2y-x-(2x-y)=3(y-x)$.

6. Dividend $=(2y-x)^{2n+1}-(y-2x)^{2n+1}$, which is divisible by $2y-x-(y-2x)=y+x$.

7. Expression is divisible by $my-nx-(mx-ny)=(m+n)y+(m+n)x=(m+n)(x+y)$.

8 Dividend $=\{(x+y)^2\}^3+\{(x-y)^2\}^3=\{(x+y)^2\}^3-\{-(x-y)^2\}^3$, which is divisible by $(x+y)^2-\{-(x-y)^2\}$ $=2(x^2+y^2)$.

9. Dividend $=(x^2+xy+y^2)^3-(-x^2+xy-y^2)^3$, which is divisible by $x^2+xy+y^2-(-x^2+xy-y^2)=2(x^2+y^2)$.

10 Dividend $=\{(a+b)^3\}^3-\{(a-b)^3\}^3$, which is divisible by $(a+b)^3-(a-b)^3=2b(3a^2+b^2)$.

11. Dividend $=(x^2+5bx+b^2)^7-(-x^2+bx-b^2)^7$, which is divisible by $x^2+5bx+b^2-(-x^2+bx-b^2)=2x^2+4bx+2b^2=2(x+b)^2$.

12. Dividend $=\{(a+b)^2\}^{2n+1}-\{-(a-b)^2\}^{2n+1}$, which is divisible by $(a+b)^2-\{-(a-b)^2\}=2(a^2+b^2)$.

13. Dividend $=\{x^3+3xy(x-y)-y^3\}^3$
$-\{-x^3+9xy(x-y)+y^3\}^3$, which is divisible by $x^3+3xy(x-y)$
$-y^3-\{-x^3+9xy(x-y)+y^3\}=2x^3-6xy(x-y)-2y^3=2(x-y)^3$.

14. By Ex. 14, Exercise XVII., the remainder $=3-5+4-2$ $=0$.

15. Let the polynome be $ax^n+bx^{n-1}+\ldots\ldots+hx+k$.
Then $a+b+\ldots\ldots+h+k=0$; subtract, term by term,
$\therefore$ $a(x^n-1)+b(x^{n-1}-1)+\ldots\ldots h(x-1)$, which is exactly divisible by $x-1$.

16. Let the polynome be $ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots+hx+k$. If this be divisible by $x+1$, it must vanish for $x=-1$, *i.e.*,
$a(-1)^n+b(-1)^{n-1}+c(-1)^{n-1}+\ldots\ldots+h(-1)=0$;
if n is even we have $\therefore$ $a-b+c-\ldots\ldots-h+k=0$,
i.e., $a+c+\ldots\ldots+k=b+$&c., and similarly when n is odd.

Exercise xix., page 43.

1. Put $a=0$ in the expn. and it becomes $-b^3y^3+b^3x^3-b^3x^3+b^3y^3=0$; $\therefore$ it is divisible by a, and $\therefore$ by b (by symmetry); put $x=0$, expn. becomes $-b^3y^3-a^3y^3+a^3y^3+b^3y^3=0$, $\therefore x$ is a factor, &c.; $\therefore$ by symmetry y is a factor. Again, put $-b$ for a, $\therefore (-bx-by)^3+(bx+by)^3+0=0$; $\therefore a+b$ is a factor. Lastly, put $x=y$; $\therefore (ay-by)^3+(by-ay)^3+0=0$, $\therefore x-y$ is a factor.

2. Substitute ax for b, $\therefore ax^3-(a^2+ax)x^2+a^2x^2=0$.

3. Substitute $-y$ for x, $\therefore (by-ay)^2-(a-b)(z-y)(by-ay)-$
$$(a-b)^2yz=(a-b)^2(y^2+yz-y^2-yz)=0;$$
$$(-ay-by)^2-(a+b)(z-y)(-ay-by)-(a+b)^2yz$$
$$=(a+b)^2(y^2+yz-y^2-yz)=0.$$

4. Putting $y=2ax$ in given expression, it becomes $6a^3x-4ax^3-20a^2x^2-6a^3x+4ax^3+20a^2x^2=0$.

6. Dividend $=x^2(x^6+y)+y^2(x^6+y)$, which is divisible by x^6+y.

7. Dividend $=a^2(c-d)+6ab(c-d)+9b^2(c-d)=(c-d)(a^2+6ab+9b^2)=(c-d)(a+3b)^2$ which, &c.

8. Put y for x, then $y\left(\frac{11}{12}y\right)^5+y\left(-\frac{11}{12}y\right)^5=0$.

9. Put b for a, then $b(b+2b)^3-b(b+2b)^3=0$; also put $-b$ for a, then $-b(-b+2b)^3-b(b-2b)^3=-b(b)^3-b(-b)^3=0$.

10. Put b for a, expression vanishes, $\therefore a-b$ is a divisor; put $-a$ for x, &c. *Or*, dividend $=a^3(a^2-2ab+b^2)+x^3(a^2-2ab+b^2)=(a^3+x^3)(a-b)^2$, which vanishes when $a-b$ does, and also when $a+x$ does.

11. Put $a=b$ in the expression, then $b(b-c)^3+b(c-b)^3+c(b-b)^3=b(b-c)^3-b(b-c)^3=0$, $\therefore a-b$ is divisor, and by symmetry $b-c$, $c-a$ are divisors.

12. Put $a=b$, then $b^3(b-c)+b^3(c-b)+c^3(b-b)=b^3(b-c)-b^3(b-c)=0$, $\therefore a-b$ is a divisor, and by symmetry $b-c$, $c-a$ are divisors.

13. Put b for a, then $b^4(b-c)+b^4(c-b)+c^4(b-b)=b^4(b-c)-b^4(b-c)=0$, $\therefore a-b$, $b-c$, $c-a$ are divisors.

14. Put $a=b$, then $0+(b-c)^2(d-b)^2-(d-b)^2(b-c)^2=0$, and so for the other divisors.

15. Dividend is symmetrical with respect to a, b, and c; put b for a, then $\{(b-c)^2+(c-b)^2\}\{(b-c)^2b^2+(c-b)^2b^2\}-\{(b-c)^2b+(c-b)^2b\}^2=b^2\{(b-c)^2+(c-b)^2\}^2-b^2\{(b-c)^2+(c-b)^2\}^2=0$; $\therefore$ $a-b$ is a divisor, and by symmetry $b-c$, $c-a$, also &c.

16. Put $-z$ for $x+y$, $-x$ for $y+z$, $-y$ for $z+x$, and expn. becomes $-xyz+xyz$ which $=0$.

17. Substitute $-a$ for $b+c$, $-b$ for $c+a$, and $-c$ for $a+b$, $\therefore$ $-abc(a-b)-abc(b-c)-abc(c-a)=-abc(a-b+b-c+c-a)=0$.

18. Square the trinomial and reduce, $\therefore$ $-2abc(b-c+a)$, &c.

19. For $a+2b$ put $3c$, for $2b-3c$ put $-a$, and for $3c-a$ put $2b$ then $27c^3-a^3-8b^3+a^3+8b^3-27c^3=0$.

20. For ab substitute $-(bc+ca)$, then

$$-(bc+ca)^3+b^3c^3+c^3a^3+3abc^2(bc+ca)$$
$$=-(bc+ca)^3+(bc+ca)^3=0.$$

Or expression is of the form $x^3+y^3+z^3-3xyz$, which is divisible by $x+y+z$.

Exercise xx., page 45.

1. 3.	2. 1.	3. $-1\pm2\sqrt{-2}$.	4. 2.	5. 36.	6. 11.
7. $-1\frac{13}{27}$.	8. -2.	9. 3.	10. 3.	11. -9.	12. 42.
13. $p=-q$, $q=6$.			14. $p=-46$, $q=14$.		

Hints and Solutions.

1. Find remainder as in Ex. 1 on dividing by x^2-x+1.

2. Find remainder on dividing by x^2-2x-1.

3. On dividing by x^2-2x+3, as in Ex. 1, remainder is found to be $-2x+1$; but $(x-1)^2=-2$, $\therefore$ $x-1=\pm\sqrt{(-2)}$ and $x=1\pm\sqrt{(-2)}$, substitute in remainder.

4. Dividend is $3x^6+11x^5+10x^3+7x^2+2x+3$, divisor x^3+3x^2-2x+5; remainder is 2.

5.

	6	9	0	−16	−5	−12	−6	60
−1				−2	−3		+6	
+4					+8	+12		−24
3	2	3	0	−6;	0	0	0	36.

remembering to "skip" two places on account of the two zero coefficients of the divisor.

6.

	1	+13	+26	+52	$+8c$
−11		−11	−22	−44	−88
	1	+2	+4	+8;	$8c-88$,

which remainder must $=0$, $\therefore c=11$.

7. Since $3x+7=0$, $\therefore x=-\frac{7}{3}$ substitute this value of x in the term $2cx$ (as in second method, page 45), and multiply (as on page 28) to avoid fractions, thus

	1	−2	−9	+0	$-\frac{14c}{3}$	−14
	1	3	9	27	81	81
	1	−6	−9	0	$378c$	−1134
−7						
	1	−13	+10	−70;	$-378c$	−644

which remainder $=0$, $\therefore c=-1\frac{19}{27}$.

8.

	1	−4	−1	+16	$+6c$:	Divide by x^2-x-6,
+1		1	−3	2		
+6			6	−18	12	
	1	−3	+2;	0	$6c+12$	$\therefore c=-1$.

9.

	2	0	−10	$4c$	6;	divide by x^2-3x+3,
+3		6	18	6		
−3			−6	−18	−6	
	2	6	2;	$4c$	−12	$\therefore c=3$.

10. $2x=5$, $\therefore x=\frac{5}{2}$ and $11x=27\frac{1}{2}$; also $-7cx^2=-43\frac{3}{4}c$; then substituting for $-7cx^2$ and $11x$, the dividend becomes 2 1 0 0 $-43\frac{3}{4}c+37\frac{1}{2}$, the multiplying to avoid fractions, as in 7 above, this becomes

	2	1	0	0	$-700c$	600
5		10	60	300	1500	
	2	12	60	300;	$700c$	−2100, $\therefore c=3$.

11. From the given condition, $cx^2 = 2\frac{1}{2}cx - 7\frac{1}{2}c$, $\therefore$ dividend becomes 4 0 0 $(110+2\frac{1}{2}c)$ $-7\frac{1}{2}c-105$; and multiplying as in last example.

$$\begin{array}{r|rrrrrr}
 & 4 & 0 & 0 & (880+20c) & -120c & -1680 \\
+5 & & 20 & 100 & -100 & & \\
-30 & & & -120 & -600 & +600 & \\
\hline
 & 4 & 20 & -20; & 180+20c, & \multicolumn{2}{r}{-120c-1080,} \quad \therefore c=-9.
\end{array}$$

See page 28. *Or* omitting c. as suggested in second method, page 45:—

$$\begin{array}{r|rrrrr}
5 & 4 & 0 & 0 & 880 & -1680 \\
-30 & & 20 & 100 & -100 & +\ 600 \\
 & & & -120 & -600 & \\
\hline
 & 4 & 20 & -\ 20; & 180, & -1080
\end{array}$$

in which remainder we have to insert $8\times 2\frac{1}{2}c$ with 180, and $-16\times 7\frac{1}{2}c$ with -1080, getting the same results as before.

12. $x^2=3x-4$, $\therefore x^3=3x^2-4x=9x-12-4x=5x-12$, $\therefore cx^3 = 5cx-12c$; so $-5x^2=15x-20$; substituting values in the dividend it becomes

$$\begin{array}{r|rrrrrr}
 & 3 & -16 & 0 & 0 & 5c-129 & -12c\ +220 \\
+3 & & 9 & -21 & -99 & -213 & \\
-4 & & & -12 & +28 & +132 & +284 \\
\hline
 & 3 & -\ 7 & -33 & -71; & 5c-210, & -12c\ +504 \quad \therefore c=42.
\end{array}$$

13. $x^2=3x-3$, $\therefore 10x^2=30x-30$, substituting this value in dividend we get

$$\begin{array}{r|rrrrr}
 & 1 & 2 & 0 & -(p+30) & q+30 \\
+3 & & 3 & 15 & 36 & \\
-3 & & & -\ 3 & -15 & -36 \\
\hline
 & 1 & 5 & 12; & -p-9, & q-6
\end{array}$$

$\therefore\ -p-9=0,\ q-6=0$, or $p=-9,\ q=6$.

14. The divisor is a^4-5a^2+7, and "skipping" for the zero coefficients in dividend and divisor, the work will stand:

$$\begin{array}{r|rrrrrrr}
 & 1 & -5 & 10 & -15 & 29 & -p & q \\
+5 & & 5 & & 15 & & 40 & \\
-7 & & & -\ 7 & & -21 & & -56 \\
\hline
 & 1 & & 3 & & 8; & 40-p, & q-56
\end{array}$$

$\therefore\ p=40,\ q=56.$

Exercise xxi., page 51

ANSWERS.

1. $b=-3,\ c=8\frac{2}{3},\ d=-24.$ 2. $c=-20\frac{1}{4},\ d=-13\frac{1}{2},\ e=60\frac{3}{4}.$

3. $b=-3,\ c=-10.$ 4. $a=3,\ b=0,\ c=-57.$

5. $a=-2,\ c=24\frac{1}{2},\ e=0.$ 6. $c=-106\frac{1}{2},\ d=202\frac{1}{2}.$

7. $a=200,\ b=-810,\ c=639.$

8. $a=4,\ c=-27,\ d=7,\ e=30.$ 9. 399.

10. $x^3-(p+3)x^2+(2p+q+3)x-(p+q+r+1).$

11. $x^3-(p-3)x^2-(2p-q-3)x-(p-q+r-1).$

12. $rx^3-(3r-q)x^2+(3r-2q+p)x-(r-q+p-1).$

13. $x^3-qx^2+prx-r^2.$ 14. $x^3-(p^2-2q)x^2+(q^2-2pr)x-r^2.$

15. $x^3-2qx^2+(pr+q^2)x+r^2-pqr.$

16. $rx^3-(pq+3r)x^2+(p^3-2pq+3r)x-(pq-r).$

33. $-1.$ 34. 1. 35. $-1.$ 36. 1. 37. $-1.$

38. $a+b+c+d.$ 39. $a+b+c+d.$ 40. $-16.$

42. Let a and c be the lengths of the parallel sides, and h and k be the lengths of the diagonals, the square of the area

$$=\tfrac{1}{16}(h+k+a+c)(h+k-a-c)(a+c+h-k)(a+c-h+k).$$

43. Let a, b, c and d be the lengths of the sides, the required polynome will be

$$\tfrac{1}{16}(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d).$$

Hints and Solutions.

2. $x^4+cx^2+dx+e=(x-1\frac{1}{2})(x+3)(x-4\frac{1}{2})(x+r)$. Equate the coefficients of x^3, $\therefore\ 0=-1\frac{1}{2}+3-4\frac{1}{2}+r$, $\therefore\ r=3$. For r substitute this value and expand. (The factor $x+r$ must be introduced into the right-hand member to raise it to the same dimensions as the left-hand member. In the whole of this Exercise the sign $=$ must be understood as the symbol of *identity.)*

3. $x^3+bx^2+cx+24=(x-2)(x+3)(x-r)$. Equating terms independent of x, $24=6r$, $\therefore\ r=4$.

4. $ax^3+bx^2+cx+90=a(x-3)(x+5)(x-2)$; $\therefore\ 90=30a$; $\therefore\ a=3$.

5. $ax^4+cx^2-30x+e=a(x-1\frac{1}{2})(x+4)(x-2\frac{1}{2})(x+r)$. Terms in x^3; $0=a(-1\frac{1}{2}+4-2\frac{1}{2}+r)$ $\therefore\ 0=-1\frac{1}{2}+4-2\frac{1}{2}+r$ $\therefore\ r=0$.

Also $e=a(-1\frac{1}{2})(4)(-2\frac{1}{2})r=0$: $-30=a(-1\frac{1}{2})(4)(-2\frac{1}{2})=15$ $\therefore\ a=-2$.

6. $81x^4+6cx^2+4dx+e=81(x-1\frac{2}{3})(x+3\frac{1}{3})(x-1\frac{1}{3})(x-r)$ $\therefore\ 0=-1\frac{2}{3}+3\frac{1}{3}-1\frac{1}{3}-r$ $\therefore\ r=\frac{1}{3}$.

7. $ax^4+bx^3+cx^2-81=a(x-\frac{3}{5})(x-\frac{3}{4})(x-3)(x+r)$. Terms in x; $0=a\{-\frac{27}{20}+(\frac{9}{20}+\frac{9}{5}+\frac{9}{4})r\}$ $\therefore\ r=\frac{3}{10}$.
Also $81=\frac{3}{5}\cdot\frac{3}{4}\cdot 3\cdot\frac{3}{10}a$ $\therefore\ a=200$.

8. $ax^4+cx^2+dx+e=a(x-2)(x-1\frac{1}{2})(x+1)(x+r)$
$\therefore\ 0=a(-2-1\frac{1}{2}+1+r)$ $\therefore\ r=2\frac{1}{2}$
$\therefore\ 14=a(1-2)(1-1\frac{1}{2})(1+1)(1+2\frac{1}{2})=3\frac{1}{2}a$ $\therefore\ a=4$.

9. $ax^3+cx+d=a(x-1\frac{1}{4})(x-2\frac{3}{4})(x+r)$
$\therefore\ 0=a(-1\frac{1}{4}-2\frac{3}{4}+r)$ $\therefore\ r=4$.
$\therefore\ 49=a(3-1\frac{1}{4})(3-2\frac{3}{4})(3+4)=\frac{49}{16}a$ $\therefore\ a=16$
$\therefore$ value $=16(-3-1\frac{1}{4})(-3-2\frac{3}{4})(-3+4)=391$.

10. $\therefore\ x-1=a$ or b or c.
$\therefore\ (x-1)^3-p(x-1)^2+q(x-1)-r$.

$$\begin{array}{r|llll}
 & 1 & -p & q & -r \\
\hline
-1 & 1 & -p-1 & q+p+1; & -r-q-p-1. \\
 & 1 & -p-2; & q+2p+3 & \\
 & 1; & -p-3 & & \\
 & 1 & & &
\end{array}$$

$$x^3 \quad -(p+3)x^2+(q+2p+3)x-(r+q+p+1).$$

11. $\therefore\ x+1=a$ or b or c
$\therefore\ (x+1)^3-p(x+1)^2+q(x+1)-r$.

12. $\therefore\ \dfrac{1}{1-x}=a$ or b or c,

$\therefore\ \left(\dfrac{1}{1-x}\right)^3-p\left(\dfrac{1}{1-x}\right)^2+q\left(\dfrac{1}{1-x}\right)-r$. Multiplying by $-(x-1)^3$ this becomes $r(x-1)^3+q(x-1)^2+p(x-1)+1$.

$$\begin{array}{r|llll} & r & q & p & 1 \\ \hline -1 & & & & \\ & r & q-r & p-q+r; & 1-p+q-r. \\ & r & q-2r; & p-2q+3r & \\ & r; & q-3r & & \\ & r & & & \end{array}$$

$$rx^3+(q-3r)x^2+(p-2q+3r)x+(1-p+q-r).$$

We have solved this problem at length from the data and by the methods used in the preceding problems, but the work might have been much shortened by taking for granted the following theorems which the student should note:

If x^3-px^2+qx-r vanish for $x=a$, or b or c

x^3+px^2+qx+r will vanish for $x=-a$, or $-b$ or $-c$

and rx^3-qx^2+px-1 will vanish for $x=\dfrac{1}{a}$ or $\dfrac{1}{6}$ or $\dfrac{1}{c}$.

Therefore rx^3+qx^2+px+1 will vanish for $x=-\dfrac{1}{a}$ or $-\dfrac{1}{b}$ or $-\dfrac{1}{c}$.

Combining this with the solution of No. 10, we get at once the solution given above.

13. $\therefore\ x=\dfrac{r}{c}$ or $\dfrac{r}{a}$ or $\dfrac{r}{b}$; or $\dfrac{x}{r}=\dfrac{1}{a}$ or $\dfrac{1}{b}$ or $\dfrac{1}{c}$

$\therefore\ r\left(\dfrac{x}{r}\right)^3-q\left(\dfrac{x}{r}\right)^2+p\left(\dfrac{x}{r}\right)-1.$

or $x^3-qx^2+prx-r^2$.

14. $\therefore\ \sqrt{x}=a$ or b or c or $\sqrt{x}=-a$ or $-b$ or $-c$

$\therefore\ (x+q)\sqrt{x}-(px+r)$ or $(x+q)\sqrt{x}+(px+r)$

$\therefore$ the required polynome $=\{(x+q)\sqrt{x}-(px+r)\}\times\{(x+q)\sqrt{x}+(px+r)\}=(x+q)^2x-(px+r)^2=x^3-(p-2q)x^2+(q^2-2pr)x-r^2$.

15. $p=a+b+c,\ q=ab+bc+ca,\ r=abc.$

$\therefore\ a(b+c)=q-bc=q-\dfrac{r}{a}$

$\therefore\ \dfrac{x-q}{r}=-\dfrac{1}{a}$ or $-\dfrac{1}{b}$ or $-\dfrac{1}{c}$.

$\therefore\ r\left(\dfrac{x-q}{r}\right)^3+q\left(\dfrac{x-q}{r}\right)^2+p\left(\dfrac{x-q}{r}\right)+1.$ See note to No. 12, above.

$\therefore\ (x-q)^3+q(x-q)^2+pr(x-q)+r^2.$

16. $\therefore\ \dfrac{x+1}{p}=\dfrac{1}{a}$ or $\dfrac{1}{b}$ or $\dfrac{1}{c}$;

$\therefore r\left(\dfrac{x+1}{p}\right)^3-q\left(\dfrac{x+1}{p}\right)^2+p\left(\dfrac{x+1}{p}\right)-1$

$\therefore r(x+1)^3-pq(x+1)^2+p^3(x+1)-p^3.$

17. $(x-1)^{12}-x^6=(x^2-2x+1)^6-x^6$

$=$ Mult. of $\{(x^2-2x+1)+x\}$ or x^2-x+1

$\therefore\ (x-1)^{12}-x^6+(x^2-x-1)=$ Mult. (x^2-x+1).

Also $(x-1)^{12}-x^6+(x^2-x+1)^2$ vanishes for $x=1$

and $\therefore\ =$ Mult. $(x-1)$.

But x^2-x+1 does not vanish for $x=1$,

and $\therefore$ it is not a Mult. $(x-1)$

$\therefore\ (x-1)^{12}-x^6+(x^2-x+1)^2=$ Mult. $(x^2-x+1)(x-1)$.

or x^3-2x^2+2x-1.

18. $(x-1)^{16}-x^8=(x^2-2x+1)^8-x^8$

$=$ Mult. $\{(x^2-2x+1)-x\}$ &c.

19. $(x-2)^{10}(2x-5)^{10}-x^{10}=(2x^2-9x+10)^{10}-x^{10}$

$=$ Mult. $\{(2x^2-9x+10)-x\}$ or $2(x^2-4x+5)$

$\therefore\ (x-2)^{10}(2x-5)^{10}-x^{10}+2^{10}(x^2-4x+5)^5$

$=$ Mult. (x^2-4x+5).

Also, $2^{10}(x^2-4x+5)^5-x^{10}=4^5(x^2-4x+5)^5-(x^2)^5$

$=$ Mult. $\{4(x^2-4x+5)-x^2\}$ or $3x^2-16x+20$, which

$=(x-2)(3x-10)$

$\therefore\ 2^{10}(x-4x+5)^5 - x^{10} = \text{Mult. } (x-2)$

$\therefore\ (x-2)^{10}(2x-5)^{10} - x^{10} + 2^{10}(x-4x+5)^5 = \text{Mult. } (x-2).$

[Instead of the part from *Also*, we might have proceeded by enquiring whether the expression would vanish for $x-2=0$ or $2x-5=0$, the factors of the first term successively equated to zero.]

x^2-4x+5 and $x-2$ are prime to each other, $\therefore$ since each measures the given expression, their product will do so or

$(x-2)^{10}(2x-5)^{10} - x^{10} + 2(x^2-4x+5)^5$

$= \text{Mult.}(x-2)(x^2-4x+5)$ or $x^3-6x^2+13x-10.$

20. $(x^2+4x+3)^{18} - x^{18}$

$= \text{Mult.}\{(x^2+4x+3)+x\}$ or x^2+5x+3

$\therefore\ (x^2+4x+3)^{18} - x^{18} - x^2 - 5x - 3 = \text{Mult.}(x^2+5x+3).$

The factors of x^2+4x+3 are $x+1$ and $x+3$. Trying these we find

$$x^{18}+x^2+5x+3=0 \text{ if } x+1=0,$$

$\therefore\ (x^2+4x+3)^{18} - x^{18} - x^2 - 5x - 3 = \text{Mult.}(x+1)(x^2+5x+3).$

21. $(9x-4)^{21}(x-1)^{21} - x^{21}$

$= \text{Mult. } \{(9x-4)(x-1)-x\}$ or $9x^2-14x+4.$

$\therefore\ (9x-4)^{21}(x-1)^{21} - x^{21} - (9x^2-14x+4)^{21}$

$= \text{Mult. } (9x^2-14x+4).$

Also $x^{21} + (9x^2-14x+4)^{21}$

$= \text{Mult. } \{x+(9x^2-14x+4)\}$ or $(9x-4)(x-1)$

$\therefore\ (9x-4)^{21}(x-1)^{21} - x^{21} - (9x^2-14x+4)^{21}$

$= \text{Mult. } (9x-4)(x-1).$

Again $(9x-4)(x-1)^{21} - (9x^2-14x+4)^{21}$

$= \text{Mult. } \{(9x-4)(x-1)-(9x^2-14x+4)\}$ or x

$\therefore\ (9x-4)^{21}(x-1)^{21} - x^{21} - (9x^2-14x+4)^{21} = \text{Mult. } x$

$\therefore\ (9x-4)^{21}(x-1)^{21} - x^{21} - (9x^2-14x+4)^{21}$

$= \text{Mult. } x(x-1)(9x-4)(9x^2-14x+4).$

22. $\{6(x-1)\}^{13}-(2x^2+3x-4)^{13}$

$=$ Mult. $\{6(x-1)-(2x^2+3x-4)\}$ or $-(2x^2-3x+2)$;

$\{6(x-1)\}^{13}+(2x^2-3x+2)^{13}$

$=$ Mult. $\{6(x-1)+(2x^2-3x+2)\}$ or $2x^2+3x-4$;

$(2x^2+3x-4)^{13}-(2x^2-3x+2)^{13}=$

Mult. $\{(2x^2+3x-4)-(2x^2-3x+2)\}$ or $6(x-1)$;

$\therefore$ given polynome $=$ Mult. of $6(x-1)(2x^2+3x-4)(2x^2-3x+2)$

23. As in 22, 1st and 2nd taken together are found to be mult. of 3rd; 2nd and 4rd taken together are found to be a mult. of 1st.; and 3rd and 1st are found to be mult. of 2nd.

24. $(2x^2+3x-4)^{16}-\{6(x-1)\}^{16}$

$=$ Mult. $\{(2x^2+3x-4)-6(x-1)\}$ or $(2x^2-3x+2)$

$(2x^2-3x+2)^{16}-\{6(x-1)\}^{16}$

$=$ Mult. $\{(2x^2-3x+2)+6(x-1)\}$ or $(2x^2+3x-4)$

and the given polynome vanishes if $x-1=0$, $\therefore$ &c.

[Problems 17 to 25 are merely cases of the following simple theorems

$$(u+v)^{2n+1}-u^{2n+1}-v^{2n+1}=\text{Mult. } uv(u+v);$$
$$(u-v)^{2n}-u^{2n}-v^{2n}+2u^mv^{2n-m}=\text{Mult. } uv(u-v.)]$$

26. If $x^2+x+1=0$. (See Example 10, p. 49, Hand-Book).

$\therefore$ $x^3=1$, $x^4=x$, $x^8=x^2$

$\therefore$ $x^8+x^4+1=x^2+x+1=0$,

i.e. $x^8+x^4+1=0$ if $x^2+x+1=0$

$\therefore$ $x^8+x^4+1=$ Mult. (x^2+x+1),

27. If $x^2+xy+y^2=0$, $\therefore x^3+x^2y+xy^2=0$,

$\therefore$ $x^3+x^2y+xy^2+y^3=y^3$, $\therefore$ $x^3+y(x^2+xy+y^2)=y^3$,

$\therefore$ $x^3=y^3$, $\therefore$ $x^9=y^9$, $\therefore$ $x^{10}=xy^9$ and $x^5=x^2y^3$

$\therefore$ $x^5y^5=x^2y^8$ $\therefore$ $x^{10}+x^5y^5+y^{10}=xy^9+x^2y^8+y^{10}$

$=y^8(xy+x^2+y^2)=0$, $\therefore$ $x^{10}+x^5y^5+y^{10}=0$

if $x^2+xy+y^2=0$ or $x^{10}+x^5y^5+y^{10}=$ Mult. (x^2+xy+y^2).

28. If $x^4+x^3+x^2+x+1=0$ $\therefore$ $x^5=1$ $\therefore$ $x^{12}=x^2$, $x^9=x^4$, $x^6=x$ $\therefore$ $x^{12}+x^9+x^6+x^3+1=x^2+x^4+x+x^3+1=0$.

29. By last exercise; $x^{16}=x$, $x^{12}=x^2$, $x^8=x^3$ and $\therefore$ $x^{16}+x^{12}+x^8+x^4+1=x+x^2+x^3+x^4+1=0$ if $x^4+x^3+x^2+x+1=0$.

30. If $x^3+x^2y+xy^2+y^3=0$ $\therefore$ $x^4=y^4$ $\therefore$ $x^{15}=x^3y^{12}$, $x^{10}y^5=x^2y^{13}$, $x^5y^{10}=xy^{14}$
$\therefore$ $x^{15}+x^{10}y^5+x^5y^{10}+y^{15}=(x^3+x^2y+xy^2+y^3)y^{12}=0$.

31. May be written $x^{17}-x^2+x^4+x^3+x^2+x+1$ $=\text{Mult. }(x^4+x^3+x^2+x+1)$. $x^{17}-x^2=x^2(x^{15}-1)$ $=\text{Mult. }(x^5-1)=\text{Mult. }(x^4+x^3+x^2+x+1)$.

32. $1+x+x^2+x^3+x^4+x^5+x^6+x^{53}-x^4$ $=\text{Mult.}(1+x+x^2+x^3+x^4+x^5+x^6)$. $x^{53}-x^4$ $=x^4(x^{49}-1)=\text{Mult.}(x^7-1)$ $=\text{Mult. }(x^6+x^5+x^4+x^3+x^2+x+1)$.

[31 and 32 may also be solved by the method employed above to solve Nos. 26 to 30. See also the note to the solution of Prob. 2, page 13. Probs. 26 to 32 may also be solved by the methods illustrated in Art. XII. of the Hand-Book. The solutions of 28 and 29 by the second of these methods will be

28. $$\frac{x^{12}+x^9+x^6+x^3+1}{x^4+x^3+x^2+x+1}=\frac{(x^{15}-1)(x-1)}{(x^5-1)(x^3-1)}$$
$$=\frac{(x^{15}-1)\{(x^6-1)-x(x^5-1)\}}{(x^5-1)(x^3-1)}=\frac{(x^{15}-1)(x^6-1)}{(x^5-1)(x^3-1)}$$
$$-\frac{(x^{15}-1)(x^5-1)x}{(x^3-1)(x^5-1)}.$$

29. $$\frac{x^{16}+x^{12}+x^8+x^4+1}{x^4+x^3+x^2+x+1}=\frac{(x^{20}-1)(x-1)}{(x^5-1)(x^4-1)}$$
$$=\frac{(x^{20}-1)\{(x^{16}-1)-x(x^{15}-1)\}}{(x^5-1)(x^4-1)}$$
$$=\frac{(x^{20}-1)(x^{16}-1)}{(x^5-1)(x^4-1)}-\frac{(x^{20}-1)(x^{15}-1)x}{(x^4-1)(x^5-1)}].$$

33 – 37 and 40. If $b-c=0$, these polynomes vanish, hence they are multiples of $b-c$, and $\therefore$ by symmetry, of D. But they are only of the same dimensions as D, viz., 6, $\therefore$ they are numerical multiples of D. The numerical multipliers may be determined, as is done in Ex. 9, page 48, Hand-Book, by assuming particular values for the letters, or it may be determined by dividing the terms involving a^3b^2c in the several polynomes by $-a^3b^2c$, which is the term in D involving these letters to these powers. Thus in 33 put $d=0$ (since d is not involved in a^3b^2c) and we obtain at once $+a^3b^2c+$&c ; and $+a^3b^2c \div -a^3b^2c=-1$.

38. This polynome is symmetrical with respect to all the letters it involves, and it vanishes if $b-c=0$, hence it either vanishes identically or it is a numerical multiple of $(a+b+c+d)D$, for a numerical multiple of $(a+b+c+d)$ is the only linear symmetrical function of a, b, c and d, and the polynome is but one degree higher than D. Assume values of a, b, c, d that will not make $(a+b+c+d)D$ vanish, and the numerical multiplier will be found to be 1. Or divide the term involving a^4b^2d in the polynome by the corresponding term in $(a+b+c+d)D$.

39. The above solution of 38 will apply, word for word, to 39.

41. The sum of the fractions $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$ ...... $\frac{1}{n}$ increased by the sum of their products two by two, increased &c.

$$=\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\ldots\left(1+\frac{1}{n}\right)-1 \text{ which}$$

$$=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\ldots\frac{n+1}{n}-1=(n+1)-1=\boldsymbol{n}.$$

42. The area will vanish if the parallel sides lie in the same straight line, in which case $h+k-a-c=0$, or $a+c+h-k$, or $a+c-h+k=0$. The area will also vanish if the diagonals (and consequently the sides) vanish simultaneously, *i.e.*, if $h+k+a+c=0$. Hence the area, and therefore its square or any other power, will involve $h+k+a+c$, $h+k-a-c$, $a+c+h-k$, $a+c-h+k$ as factors. But it is given that the square of the area is a polynome

of four dimensions, hence it can only be a numerical multiple of these four factors. Write s^2 for the square of the area and m for a numerical constant

$$s^2 = m(h+k+a+c)(h+k-a-c)(a+c+h-k)(a+c-h+k).$$

To determine m, let $c=0$ in which case the trapezium becomes a triangle with sides a, h, k and by Ex. 6, p. 47, Hand-Book.

$$s^2 = \tfrac{1}{16}(h+k+a)(h+k-a)(a+h-k)(a-h+k),$$

$$\therefore\ m = \tfrac{1}{16}.$$

Or m may be determined by taking as a particular case a rectangle with adjacent sides of lengths 3 and 4, in which case $a=c=4$, $h=k=5$, and $s=12$, from which it will be found $m=\frac{1}{16}$.

[The formula for the area of any quadrilateral is

$$16s^2 = (2hk+a^2-b^2+c^2-d^2)(2hk-a^2+b^2-c^2+d^2)$$

in which s is the area, h and k the lengths of the diagonals and a, b, c, d the lengths of the sides taken in order. This formula is due to Gauss.]

43. The only way in which the area of a quadrilateral inscribed in a circle can vanish is by three of the sides becoming equal to the fourth, *i.e.*,

$$a+b+c-d=0, \text{ or } b+c+d-a=0, \text{ or } d+a+b-c=0.$$

If all the sides vanish simultaneously, it will be found to be only a particular case if one of these four, or rather of all of them simultaneously.

$$\therefore\ s^2 = m(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d).$$

To determine m take the case of the square $\therefore\ a=b=c=d$, $s=a^2$.

$$\therefore\ a^4 = m2a.2a.2a.2a.$$

$$\therefore\ m = \tfrac{1}{16}.$$

Exercise xxii., page 53.

1. Transfer the right-hand member and the resulting polynome on the left vanishes for $x=0$ or $+b$ or $-b$, *i.e.*, for three different values although it involves only the second power of x, hence it vanishes identically.

2 and 3 may be proved in like manner. In No. 3 divide through by z^2 and replace z by c; the formula will then be seen to be symmetrical with respect to a, b and c.

[These theorems are merely cases of

$$\frac{(x-a)(y-a)(z-a)}{(b-a)(c-a)} + \text{two similar terms} = x+y+z \quad a-b-c,$$

$$\frac{(x-a)(y-a)}{(b-a)(c-a)} \qquad + \quad \text{``} \qquad \text{``} \qquad \text{``} \quad = 1, \text{ and}$$

$$\frac{(x-a)(y-a)}{(b-a)(c-a)(d-a)} + \text{three} \quad \text{``} \qquad \text{``} \quad = 0$$

respectively. They may be proved like Example 1., Page 53 Hand-Book, or by the method applied to Probs. 8–25, Exercise xxiii. For the general theorem see the solutions of those problems.]

4. Change the signs of a, b, c, in Example 1., page 53, Hand-Book.

5. Holds for $a=0$, or $-b$ or $-c$ or $-(b+c)$ $\therefore$ &c. Or thus

$$bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)=$$

$$(b+c)bc(b-a)+(c+a)ca(c-a)+(a+b)ab(a-b) \qquad \text{A.}$$

Also $0=abc(b-c)+ \quad b\,ca(c-a)+ \quad c\,ab(a-b)$ B.

then $A+B$ gives, &c.

6. Treat as a polynome in a. This identity is a case of

$$\frac{a+x}{(x-y)(x-z)(x-u)} + \text{three similar terms} = 0.$$

7. $a^2(b-c)+b^2(c-a)+c^2(a-b) = -(a-b)(b-c)(c-a)$

$\therefore$ $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2) =$

$$-(a^2-b^2)(b^2-c^2)(c^2-a^2)$$

and $(a+b+c)^3-a^3-b^3-c^3 = 3(a+b)(b+c)(c+a$, by [8].

8. See Prob. 2, Exercise V.

$$(a^2+b^2)(c^2+d^2)(e^2+f^2) = \{(ad+bc)^2+(bd-ac)^2\}(e^2+f^2)$$
$$= \{(ad+bc)f+(bd-ac)e\}^2 + \{ad+bc)e-(bd-ac)f\}^2.$$

9. See Prob. 9, Exercise XI. Or thus

If $x^2+xy+y^2=0 \quad x^3=y^3$ and $x\,(x+y)=-y^2$

$\therefore\ x^5(x+y)^5=-y^{10}=-x^9y\ \therefore\ (x+y)^5=-x^4y$

$\therefore\ (x+y)^5-x^5=-x^4y-x^5-x^3y^2=-x^3(x^2+xy+y^2)=0.$

$\therefore\ (x+y)^5-x^5-y^5=0$ if $x^2+xy+y^2=0$

$\therefore\ x^2+xy+y^2$ is a factor of $(x+y)^5-x^5-y^5$.

Also $(x+y)^5-x^5-y^5=0$ if $x=0$ or $y=0$ or $x+y=0$

$$\therefore\ (y+y)^5-x^5-y^5=kxy(x+y)(x^2+xy+y^2)$$
$$=\tfrac{1}{2}kxy(x+y)\{(x+y)^2+x^2+y^2\}.$$

Determining k it will be found $=5$.

Substitute $a-b$ for x, $b-c$ for y, and $\therefore\ -(c-a)$ for $x+y$

$$(a-b)^5+(b-c)^5+(c-a)^5$$
$$=\tfrac{5}{2}(a-b)(b-c)(c-a)\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

10. The left-hand side vanishes for $x=0$, or $y=0$, or $z=0$, and it is of but three dimensions, $\therefore$ it $=kxyz$. k may now be determined. Or, transfer the left-hand member, and the resulting polynome on the left will vanish for $x=0$, or $y-z$, or $-y+z$, or $-y-z$, *i.e.*, for *four* values, hence it vanishes identically.

11. By Ex. 2, p. 18, Hand-Book,

$$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$$

and $\therefore\ (a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3$

$$=3(a^2-b^2)(b^2-c^2)(c^2-a^2).$$

12. $x^2(y+z)^2+xyz(y+z)=x(y+z)(xy+yz+zx)$

$$y^2(z+x)^2+xyz(z+x)=y(z+x)(xy+yz+zx)$$
$$z^2(x+y)^2+xyz(x+y)=z(x+y)(xy+yz+zx)$$

$\therefore$ by addition, &c.

Exercise xxiii., page 57.

ANSWERS.

1. $5b^4+15c^4$. 2. 6. 3. 3.

4. $-\{A(b+c+d)+B(a+c+d)+C(a+b+d)+D(a+b+c)\}$.

5. 0. 6. $5b^4-30ab^3+30a^2b^2-5a^3b$. 8. 0. 9. 0.

10. 0. 11. 1. 12. $a+b+c+d$. 13. -1.

14. $a+b+c+d$. 15. $(a+b+c)(a^2+b^2+c^2+ab+bc+ca)+abc$.

16. $(a+b+c)^2(a^2+b^2+c^2)+2abc(a+b+c)$.

17. $a+b+c+d)$. 18. $(a+b+c+d)^2$.

19. $(a+b+c+d)\{(a+b+c+d)^2-(ab+bc+cd+ac+bd+ad)\}+abcd$.

20. $a+b+c$. 21. 3. 22. -1.

23. 0. 24. 0. 25. 0.

26. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$. 27. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$.

28. $1+x+x^2+x^3$. 29. $1-2x+3x^2-4x^3$.

30. $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3$.

Hints and Solutions.

7. Let $(1+ax+bx^2+cx^3+\&c.)(1-ax+bx^2-cx^3+\&c.)=$
$1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\&c.$

Change x into $-x$.

$\therefore\ (1-ax+bx^2-cx^3+\&c.)(1+ax+bx^2+cx^3+\&c.)=$
$1-Ax+Bx^2-Cx^3+Dx^4-Ex^5+\&c.$

$\therefore\ 1+Ax+Bx^2+Cx^3+\&c.=1-Ax+Bx^2-Cx^3+\&c.$

$\therefore\ Ax+Cx^3+Dx^5+\&c.=0.$

Now this is to hold for any value whatsoever that may be given to x; $\therefore\ A=0,\ C=0,\ D=0,$ &c.

8. Assume $\dfrac{1}{(x-a)(x-b)(x-c)(x-d)}=$

$$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\frac{D}{x-d} \qquad (\alpha)$$

Multiply by $(x-a)(x-b)(x-c)(x-d)$

$\therefore\ 1=(A+B+C+D)x^3-\&c.$

$\therefore\ A+B+C+D=0.$

Multiply both sides of (α) by $x-a$ and in the result put $x=a$

$$\therefore\ \frac{1}{(a-b)(a-c)(a-d)}=A.$$

$\therefore$ by symmetry, &c.

9. Assume $\dfrac{x}{(x-a)(x-b)(x-c)(x-d)}=\dfrac{A}{x-a}+\&c.$

10. Example 1, page 55, Hand-Book.

11. Assume $\dfrac{x^3}{(x-a)(x-b)(x-c)(x-d)}=\dfrac{A}{x-a}+\&c.$

12. Assume $\dfrac{x^4}{(x-a)(x-b)(x-c)(x-d)}=x-\dfrac{A}{x-a}+\&c.$

$\therefore\ x^4=x^4-(a+b+c+d-A-B-C-D)x^3+\&c.$

$\therefore\ 0=a+b+c+d-A-B-C-D$

$\therefore\ A+B+C+D=a+b+c+d.$

13. Assume $\dfrac{1}{(x-a)(x-b)(x-c)(x-d)(x-e)}=\dfrac{A}{x-a}+\&c.$

$A+B+C+D+E=0.$

Also $A=\dfrac{1}{(a-b)(a-c)(a-d)(a-e)}$, $B=\&c.$

$$\therefore \frac{1}{(a-b)(a-c)(a-d)(a-e)}+\ldots\ldots+\frac{1}{(e-a)(e-b)(e-c)(e-d)}=0$$

Let $e=0$, and multiply by $abcd$;

$$\therefore\ \frac{bcd}{(a-b)(a-c)(a-d)}+\ \ldots\ldots\ldots\ +1=0,$$

Or thus: Prove by the method employed in Ex. 1, p. 53, Hand-Book,

$$\frac{(y-b)(y-c)(y-d)}{(a-b)(a-c)(a-d)}+\ \ldots\ldots\ldots\ =1.$$

Then take the case $y=0$.

14. *1st Solution.* Find a polynome in x that will become $a(a+b)(a+c)$, $b(b+c)(b+a)$, or $c(c+a)(c+b)$ according as $x=a$ or b or c.

$a(a+b)(a+c)=\frac{1}{2}(a+a)(a+b)(a+c)$

$b(b+c)(b+a)=\frac{1}{2}(b+a)(b+b)(b+c)$

$c(c+a)(c+b)=\frac{1}{2}(c+a)(c+b)(c+c)$

$\therefore$ required polynome $=\frac{1}{2}(x+a)(x+b)(x+c)$.

Assume $\dfrac{(x+a)(x+b)(x+c)}{(x-a)(x-b)(x-c)} = 1 + \dfrac{A}{x-a} + \&c.$ (α)

$\therefore\ x^3+(a+b+c)x^2+\&c. = x^3 - \{(a+b+c)-(A+B+C)\}x^2+\&c.$

$$\therefore\ 2(a+b+c) = A+B+C.$$

Multiply (α) by $x-a$ and let $x=a$.

$\therefore\ \dfrac{2a(a+b)(a+c)}{(a-b)(a-c)} = A$, values of B and C by symmetry.

Add together the values of A, B and C thus obtained and take half.

2nd Solution. Assume $(x-a)(x-b)(x-c) = x^3+px^2+qx+r$.

$\therefore\ p = -(a+b+c),\ q = ab+bc+ca,\ r = -abc.$

Express $a(a+b)(a+c)$ in terms of a, p, q, r.

$a(a+b)(a+c) = a(a^2+ab+ac+bc) = a^2(a+b+c)+abc = -pa^2-r$

Substitute x for a and use the resulting polynome for a numerator.

$$\frac{-px^2-r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \&c.$$

$$\therefore\ -px^2-r = (A+B+C)x^2-\&c.$$

$$\therefore\ A+B+C = -p = a+b+c.$$

Determine A, B and C as before.

[Either of these solutions is easily derived from the other, for any multiple of $(x-a)(x-b)(x-c)$ may be added to or taken from any polynome satisfying the conditions stated in the first solution, and the resulting polynome will still satisfy those conditions. Now

$$\tfrac{1}{2}(x+a)(x+b)(x+c) - \tfrac{1}{2}(x-a)(x-b)(x-c) = -px^2-r,$$

showing how the second solution may be derived from the first.]

15. $a^3(a+b)(a+c) = -pa^4-ra^2$.

Assume $\dfrac{-px^4-rx^2}{(x-a)(x-b)(x-c)} = -px+p^2+\dfrac{A}{x-a}+\&c.$

$\therefore\ -px^4-rx^2 = -px^4+(p^3-pq+A+B+C)x^2-\&c.$

$\therefore\ -r = p^3-pq+A+B+C$

$\therefore\ A+B+C = -p(p^2-q)-r.$

16. $a^4(a+b)(a=c)=-pa^5-ra^3$

Assume $\dfrac{-px^5-rx^3}{(x-a)(x-b)(x-c)}=-px^2+p^2x-p^3+pq-r+\dfrac{A}{x-a}+\&c$

$\therefore\ -px^5-rx^3=-px^5-rx^3+(-p^4+2p^2q-$
$2pr+A+B+C)x^2-\&c.$

$\therefore\ A+B+C=p^4-2p^2q+2pr.$

17, 18, 19, $a(a+b)(a+c)(a+d)=$
$a\{a^3+a^2(b+c+d)+a(bc+cd+db)+bcd\}=$
$a(-pa^2-r),$

if $(x-a)(x-b)x-c)(x-d)=x^4+px^3+qx^2+rx+s$ so that
$p=-(a+b+c+d),\ q=ab+bc+cd+da+ac+bd$
$r=-(abc+bcd+cda+dab),\ s=abcd.$

For 17, assume $\dfrac{-px^3-rz}{(x-a)(x-b)(x-c)(x-d)}=\dfrac{A}{x-a}+\&c.$
giving $A+B+C+D=-p,$

For 18, assume $\dfrac{-px^4-rx^2}{(x-a)(x-b)(x-c)(x-d)}=-p+\dfrac{A}{x-a}+\&c.$
giving $A+B+C+D=p^2,$

For 19, assume $\dfrac{-px^5-rx^3}{(x-a)(x-b)(x-c)(x-d)}=-px+p^2+\dfrac{A}{x-a}+\&c.$
giving $A+B+C+D=-p^3+pq-r.$

[Divide $-px^5-rx^3$ by x^3+px^2+qx+r and compare the coefficients of the powers of x in the quotient with the values of the expression in 14–19].

20. *1st Solution.* Let $(x-a)(x-b)(x-c)=x^3+px^2+qx+r$

$$b = -(a+p)-c$$
$$c = -(a+p)-b$$
$$b+c-a=-(a+p)-a$$
$$\therefore\ bc(b+c-abc)=-\{(a+p)+a\}\{(a+p)+b\}\{(a+p)+c\}$$
$$=-(a-p)^3+p(a+p)^2-q(a+p)+r$$
$$\therefore\ -bc(b+c)=(a+p)^3-p(a+p)^2+q(a+p).$$

Substitute x for a in the right-hand member and expand (See Ex. 5, p. 47, Hand-Book), and we obtain for numerator

$$x^3+2px^2+(p^2+q)x+pq.$$

Assume $\dfrac{x^3+2px^2+(p^2+q)x+pq}{(x-a)(x-b)(x-c)}=1+\dfrac{A}{x-a}+\&c. \qquad (\alpha)$

$$\therefore\ x^3+2px^2+\&c.=x^3+(p+A+B+C)x^2+\&c.$$

$$\therefore\ A+B+C=p=-(a+b+c).$$

Mult. (α) by $x-a$ and then put $x=a$.

$$A=\frac{-bc(b+c)}{(a-b)(a-c)},\ B=\&c.$$

2nd Solution. $b+c=-(a+p)$. $bc=-a(b+c)+q=a^2+ap+q.$

$$\therefore\ bc(b+c)=-(a^2+pa+q)(a+p)$$
$$=-\{a^3+2pa^2+(p^2+q)a+pq\}.$$

The solution now proceeds as in 1st.

3rd Solution. $bc(b+c)=bc(a+b+c)-abc=-pbc+r$

$$=-p(a^2+ap+q)+r.$$

Assume $\dfrac{px^2+p^2x+pq-r}{(x-a)(x-b)(x-c)}=\dfrac{A}{x-a}+\&c.$

$$\therefore\ A+B+C=p.$$

21. *1st Solution.* $2a+b=(a-p)-c$

$$2a+c=(a-p)-b$$
$$a+b+c=(a-p)-a$$

$$\therefore (a+b+c)(2a+b)(2a+c)=(a-p)^3+p(a-p)^2+q(a-p)+r$$
$$=a^3-2pa^2+(p^2+q)a-(pq-r).$$

Assume $\dfrac{x^3-2px^2+(p^2+q)x-(pq-r)}{(x-a)(x-b)(x-c)}=1+\dfrac{A}{x-a}+\&c.$

Determine $A+B+C$, and A, B, C. (It will be found to be necessary to divide through by $a+b+c$, which was introduced as a factor.)

2nd Solution. $(2a+b)(2a+c)=4a^2+2a(b+c)+bc$

$$=3a^2+a(a+b+c)+ab+bc+ca$$
$$=3a^2-pa+q.$$

$$\therefore \text{ Assume } \frac{3x^2-px+q}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \&c.$$

$\therefore A+B+C=3.$

22. $a(b+c)=-a\{a-(a+b+c)\}=-a(a+p).$

$$\text{Assume } \frac{-x^2-px}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \&c.$$

$\therefore A+B+C=-1.$

23. $b+c+d=-(a+p)$

$$\text{Assume } \frac{-x-p}{(x-a)(x-b)(x-c)(x-d)} = \frac{A}{x-a} + \&c.$$

$\therefore A+B+C+D=0.$

Or, $(a+b+c+d)$(Problem 8) $-$ (Problem 9).

24. $bc+cd+db=-a(b+c+d)+q=a^2+pa+q,$

$$\text{Assume } \frac{x^3(x^2+px+q)}{(x-a)(x-b)(x-c)(x-d)} = x+\frac{A}{x-a}+\&c.$$

$\therefore x^5+px^4+qx^3=x^5+px^4+(q+A+B+C+D)x^3+\&c.$

$\therefore A+B+C+D=0.$

Or, $a^3(bc+cd+db)=a\{a^2(bc+cd+db)\}$

$=a\{a(abc+bcd+cda+dab)-abcd\}$

$=a\{-ra-s\}.$

$$\therefore \text{ Assume } \frac{-rx^2-s}{(x-a)(x-b)(x-c)(x-d)} = \frac{A}{x-a}+\&c.$$

$\therefore A+B+C+D=0.$

25. $bc+cd+db=a^2+pa+q$

$$\text{Assume } \frac{x^2+px+q}{(x-a)(x-b)(x-c)(x-d)} = \frac{A}{x-a}+\&c.$$

$\therefore A+B+C+D=0.$

[Probs. 8–25 are all included in the general theorem, "If there are m quantities h_1, h_2. h_3, ... h_m and n quantities (no two of which are equal) a_1, a_2, a_3, ... a_n.

$$\Sigma \frac{(a_1-h_1)(a_1-h_2)(a_1-h_3) \;.\;.\;.\;.\; (a_2-h_m)}{(a_1-a_2)(a_1-a_3)(a_1-a_4) \;.\;.\;.\;.\; (a_1-a_n)} =$$

term independent of x in quotient of

$$\{x(x-h_1)(x-h_2) \;.\;.\;.\;.\; (x-h_m)\} \div$$
$$\{(x-a_1)(x-a_2) \;.\;.\;.\;.\; (x-a_n)\}.$$

Cor. If $m+1$ is less than n, there is no 'term' and the value $=0$; if $m+1=n$, the 'term' $=1$.

The student will find it an excellent exercise to obtain Probs. 8–25 from this general theorem; thus for 8 let $m=1$ and $n=4$, and assume $h_1=a_1-1$, $a_1=a$, $a_2=b$, $a_3=c$, $a_4=d$.]

26. Assume $1+x=(1+ax+bx^2+cx^3+\&c.)^2$

$$=1+2ax+(a^2+2b)x^2+2(ab+c)x^3+\&c.$$

$\therefore \quad 2a=1 \quad \therefore a=\frac{1}{2}.$

$a^2+2b=0 \quad \therefore b=-\frac{1}{8}.$

$ab+\;c=0 \quad \therefore c=\frac{1}{16}.$

$$\therefore \sqrt{(1+x)}=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3.$$

30. Assume $1+x=(1+ax+bx^2+cx^3+\&c.)^3$

$$=1+3ax+3(a^2+b)x^2+(a^3+6ab+3c)x^3+\&c$$

$\therefore 3a=1 \quad \therefore a=\frac{1}{3}$; $a^2+b=0 \;\therefore\; b=-\frac{1}{9}$

$a^3+6ab+3c=0 \;\therefore\; 3c=\frac{6}{27}-\frac{1}{27}=\frac{5}{27}, \;\therefore\; c=\frac{5}{81}.$

Exercise xxiii. (*a*), page 61.

ANSWERS.

1. $(p-p'+q)^2=(p+1)(p^2-pp'-q).$

8. $9(p^2-q)(r^2-qt)-(pr-t)^2=$

$9\{3(p^2-q)(qr-pt)-(pq-r)(pr-t)\}\times$

$\{3(pq-r)(r^2-qt)-(pr-t)(qr-pt)\}.$

9. $\dfrac{4x^n(x^3+3px^2+3qx+r)}{x^4+4px^3+6qx^2+4rx+t}.$

10. $-4p$; $(4p)^2-2(6q)$; $-(4p)^3+3(4p)(6q)-3(4r)$; $(4p)^4-4(4p)^2(6q)+4(4p)(4r)+2(6q)^2-4t$

$-(4p)^5+5(4p)^3(6q)-5(4p)^2(4r)-5(4p)(6q)^2+5(4p)t+5(6q)(4r)$;
$(4p)^6 \quad 6(4p)^4(6q)+6(4p)^3(4r)+9(4p)^2(6q)^2-6(4p)^2t$
$-12(4p)(6q)(4r)-2(6q)^3+6(6q)t+3(4r)^2$.

11. $s_0s_4-4s_1s_3+3s_2^2$; $s_0s_6-6s_1s_5+15s_2s_4-10s_3^2$.

12. $\dfrac{x^n(4x^3-28x+1)}{x^4-14x^2+x-38}$; $s_1=0$, $s_2=28$, $s_3=-3$, $s_4=544$, $s_5=-70$, $s_6=8683$; $\Sigma(a-b)^4=4526$; $\Sigma(a-b)^6=264122$.

Hints and Solutions.

1. $U=x^2+px+q$, $V=x^3+p'x+q$.

Any common factor of U and V will be a factor of $(V-U)\div x$ $=x^2-x+p'-p$.

$\therefore$ $x^2-x+p'-p$ and x^2+px+q must have a C. F. Now apply Ex. 1, Art. XI., p. 58.

2nd Solution. Any C. F. of U and V will be a C. F. of

$$\{(x+1)U-V\}\div x=(p+1)x+(p-p'+q)$$

$$\text{and } V-(x-p)U=(p^2+p'-q)x+(p+1)q$$

$\therefore$ if U and V have a C. F.

$$\frac{p+1}{p^2+p'-q}=\frac{p-p'+q}{(p+1)q}$$

$$\text{or } (p+1)^2q=(p-p'+q)(p^2+p'-q).$$

2. If expn. = exact cube it must

$$=(x^2+a^2x+f)^3=x^6+3a^2x^5+3(a^4+f)x^4+(a^6+6a^2f)x^3+3(f^2+a^4f)x^2+3a^2f^2x+f^3.$$

Equating coefficients (Theor. IV.),

$$a^4+f=b$$
$$a^6+6a^2f=c$$
$$f^2+a^4f=d \text{ or } bf=d$$
$$a^2f^2=e^2$$
$$\therefore f=\frac{e}{a}=\frac{d}{b}=\frac{c-a^6}{6a^2}=b-a^4.$$

3. *1st Solution.* $U=ax^5+bx+c$, $V=a+bx^4+cx^5$

$$R=\{(ax+c)V-(cx+a)U\}\div x=$$
$$(c^2-a^2+ab)x^4+bcx^3-bcx-(c^2-a^2+ab)=$$
$$(c^2-a^2+ab)(x^4-1)+bc(x^2-1)x$$
$$\{(c^2-a^2+ab)(x^2+1)+bcx\}(x^2-1).$$
$$S=(cx+b)U-axV=$$
$$bcx^2+(c^2-a^2+b^2)x+bc.$$

Any quadratic C.F. of U and V must be a C.F. of R and S or, (rejecting the factor x^2-1 in R which involves a particular value of x) of

$$(c^2-a^2+ab)x^2+bcx+(c^2-a^2+ab)$$
$$\text{and } bcx^2+(c^2-a^2+b^2)x+bc;$$
$$\therefore\ \frac{c^2-a^2+ab}{bc}=\frac{bc}{c^2-a^2+b^2}=\frac{c^2-a^2+ab}{bc}$$

2nd Solution. If mx^2+nx+p is a factor of U, px^2+nx+m will be a factor of V, $\therefore$ if mx^2+nx+p is a C.F. of U and V, $m=p$ and $\therefore$ every C.F. is of the form mx^2+nx+m which may be written $m(x^2+kx+1)$.

$$(ax^5+bx+c)\div(x^2+kx+1)=ax^3-akx^2+\text{lower terms.}$$
$$(c+bx+ax^5)\div(1+kx+x^2)=c+(b-ck)x+\text{ higher terms.}$$

Assume $U=(x^2+kx+1)\{ax^3-akx^2+(b-ck)x+c\}$

i.e. $ax^5+bx+c=$

$$ax^5-(ak^2+ck-a-b)x^3-(ck^2+ak-bk-c)x^2+bx+c.$$
$$\therefore\ ak^2+ck-(a+b)=0$$
$$ck^2+(a-b)k-c=0.$$

These equations must both hold for one and the same value of k. they must $\therefore$ have a common factor, $\therefore$ by Ex. 1, Art. XI., Hand-Book,

$$b^2c^2=(c^2-a^2-ab)\{c^2-(a^2-b^2)\}.$$

4. *1st Solution.* $\{(ax+c)V-(cx+a)U\}\div x=$

$$(c^2-a^2)x^4+abx^3-abx-(c^2-a^2)=$$
$$(cx^2-a^2)(x^4-1)+ab(x^2-1)x=$$
$$\{(c^2-a^2)(x^2+1)+abx\}(x^2-1)=W.$$

Also $\{ax^3+c)V-(cx^3+a)U\}\div x^2=$

$abx^4+(c^2-a^2-bc)x^3-(c^2-a^2-bc)x-ab=$

$ab(x^4-1)+(c^2-a^2-bc)(x^2-1)x=$

$\{ab(x^2+1)+(c^2-a^2-bc)x\}(x^2-1)=W'$

$\therefore$ W and W' can differ only by a numerical factor,

$$\therefore \quad \frac{c^2-a^2}{ab}=\frac{ab}{c^2-a^2-bc}.$$

2nd Sol. $(ax^5+bx^2+c)=(x^2+kx+1)(x^3-akx^2-ck+1)=$

$ax^5-(ak^2+ck-a)x^3-(ck^2+ak-c)x^2+c$

$\therefore \ ak^2+ck-a=0$ and $-(ck^2+ak-c)=b$

i.e. $ck^2+ak+b-c=0$

$\therefore \ a^2b^2=(a^2-c^2)\{a^2+c(b-c)\}.$

5. $ax^4+bx^3+cx+d=(x^2+kx+1)\{ax^2+(b-ak)x+d\}=$

$ax^4+bx^3-(ak^2-bk-a-d)x^2-(ak-dk-b)x+d$

$\therefore \ ak^2-bk-(a+d)=0$

and $(a-d)k-(b-c)=0$

$\therefore \ dk^2-ck-(a+d)=0$

$\therefore \ (a-d)^2(a+d)^2=(bd-ac)\{b(a+d)-c(a+d\}$

$\therefore \ (a-d)^2(a+d)=(bd-ac)(b-c).$

6. $(x^3+px^2+qx+r)\div(x^2+ax+b)=$

$x-(a-p)+$linear remainder.

For exact division this remainder must vanish, and $\therefore$ $x-(a-p)$ must be a factor of x^3+px^2+qx+r, which will $\therefore$ $=0$ if in it $a-p$ be substituted for x.

$$\therefore \ (a-p)^3+p(a-p)^2+q(a-p)+r=0.$$

Expand by Horner's method, Hand-Book, Art. VIII.

$(x^3+px^2+qx+r)\div(x^2+ax+b)=$

$\left(x+\dfrac{r}{b}\right)+x$ (linear polynome)

$\therefore\ x+\dfrac{r}{b}$ must be a factor of x^3+px^2+qx+r, which will $\therefore$

$=0$ if $x=-\dfrac{r}{b}$

$$\therefore -\frac{r^3}{b^3}+p\,\frac{r^2}{b^2}-q\,\frac{r}{b}+r=0.$$

7. $(x^4+px+q)\div(x^2+ax+b)=$
$x^2-ax+(a^2-b)+$ linear remainder;

$\therefore\ (x^4+px+q)=(x^2+ax+b)(x^2-ax+a^2-b)=$
$x^4+(a^3-2ab)x+(a^2-b)b.$

$\therefore\ a^3-2ab=p.\quad (a^2-b)b=q\ \therefore\ (2a^3-2ab)2ab=4a^2q$

$\therefore\ (a^3+p)(a^3-p)=4a^2q.$ I.

Also $a^2b(a^2b-2b^2)^2=p^2b^3$

$\therefore\ (q+b^2)(q-b^2)^2=p^2b^3.$ II.

8.
$$\begin{array}{c|ccccc}
 & 1 & 4p & 6q & 4r & t \\
a & & a & a^2+4ap & a^3+4a^2p+6aq & T \\
\hline
 & 1 & a+4p & a^2+4ap+6q & a^3+4a^2p+6aq+4r & R \\
a & & a & 2a^2+4ap & 3a^3+8a^2p+6aq & \\
\hline
 & 1 & 2a+4p & 3a^2+8ap+6q\,; & 4a^3+12a^2p+12aq+4r &
\end{array}$$

$\therefore\ 4a^3+12a^2p+12aq+4r=0.$

$\therefore\ x^3+3px^2+3qx+r$ is divisible by $x-a$

and $\therefore$ has a C. F. with $px^3+3qx^2+3rx+t$.

9. $$\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\frac{1}{x-d}=$$
$$\frac{4x^3-3(a+b+c+d)x^2+2\sum abx-(abc+bcd+cda+dab)}{x^4+4px^3+6qx^2+4rx+t}=$$
$$\frac{4x^3+12px^2+12qx+4r}{x^4+4px^3+6qx^2+4rx+t}.$$

Multiply this by x^n.

10.
$$\begin{array}{l|lllll}
 & 4 & 12p & 12q & 4r & \\
-4p & & -16p & 16p^2 & -64p^3+48pq & 256p^4-288p^2q+48pr \\
-6q & & & -24q & 24pq & -96p^2q+72q^2 \\
-4r & & & & -16r & 16pr \\
-t & & & & & -4t \\
\hline
 & 4, & -4p, & 16p^2-12q, & -64p^3+72pq-12r, &
\end{array}$$

Continue the division to three more terms.

11. $(x-a)^4+(x-b)^4+(x-c)^4+(x-d)^4=$

$$4x^4-4(a+b+c+d)x^3+6(a^2+b^2+c^2+d^2)x^2$$
$$-4(a^3+b^3+c^3+d^3)x+(a^4+b^4+c^4+d^4)$$
$$=4x^4-4s_1x^3+6s_2x^2-4s_3x+s_4.$$

Put $x=a,\ =b,\ =c,\ =d$ in succession,

$$\therefore\ (a-b)^4+(a-c)^4+(a-d)^4 \quad = s_0a^4-4s_1a^3+6s_2a^2-4s_3a+s_4$$
$$(a-b)^4 \quad +(b-c)^4+(b-d)^4 = s_0b^4-4s_1b^3+6s_2b^2-4s_3b+s_4$$
$$(a-c)^4 \quad +(b-c)^4+(c-d)^4 = s_0c^4-4s_1c^3+6s_2c^2-4s_3c+s_4$$
$$(a-d)^4 \quad +(b-d)^4+(c-d)^4 = s_0d^4-4s_1d^3+6s_2d^2-4s_3d+s_4$$
$$\therefore\ 2\{(a-b)^4+(a-c)^4+\ \dots\ +(c-d)^4\}=$$
$$s_0s_4-4s_1s_3+6s_2s_2-4s_3s_1+4s_4$$
$$\therefore\ \Sigma\,(a-b)^4=s_0s_4-4s_1s_3+3s_2^2.$$

Similarly; $(x-a)^6+(x-b)^6+(x-c)6+(x-d)^6=$

$$s_0x^6-6s_1x^5+15s_2x^4-20s_3x^3+15s_4x^2-6s_5x+s_6.$$

Put $x=a,\ =b,,\ =c,\ =d$ in succession and proceed as before.

N.B. s_0, s_1, s_2, &c., are the coefficients of the terms taken in order of the quotient in No. 10; they are $\therefore$ known polynomes in p, q, r and t.

12. Substituting in Ex. 9, $p=0$, $6q=-14$, $4r=1$, $t=-38$, the result there obtained becomes

$$\frac{x^n(4x^3-28x+1)}{x^4-14x^2+x-38}.$$

Dividing as in Ex. 10, omitting the line for p.

	4	0	-28	1			
14			56		392	-42	7616
-1				-4		-28	3
38					152		1064
	4	0	28	-3	544	-70	8683

$\therefore\ s_0=4,\ s_1=0,\ s_2=28,\ s_3=-3,\ s_4=544,\ s_5=-70,\ s_6=8683.$

Now substitute these values in expanded and reduced expressions for $\Sigma\,(a-b)^4$ and $\Sigma\,(a-b)^6$.

CHAPTER III.

Exercise xxiv., page 62.

1. $(3m+2)^2$; $(c^m-1)^2$.

2. $(y^3-z^3)^2$; $4y^2(2x+y)^2$.

3. $(3ab+2c)^2$; $4y^2(3x-y)^2$.

4. $(\frac{1}{2}x^2-4yz)^2$; $(\frac{1}{2}a^2-\frac{1}{3}b^2c^2)^2$.

5. $(a+b+c)^2$; $(3x^4-\frac{1}{4}y^2)^2$.

6. $(z-x+y)^2$; $\left\{\left(\frac{a}{b}\right)^m-\left(\frac{b}{a}\right)^m\right\}^2$.

7. $(x^2-z^2)^2$.

8. $(x^2-2xy+y^2)^2=(x-y)^4$

9. $(a+b+c-c)^2=(a+b)^2$; $(\frac{3}{4}p^3-\frac{4}{3}q^3)^2$.

10. $(3x-4y-2x+3y)^2=(x-y)^2$.

11. $(x^2-xy+y^2+x^2+xy+y^2)^2=4(x^2+y^2)^2$.

12. $(5x^2+2xy+7y^2-4x^2-6y^2)^2$
$=(x^2+2xy+y^2)^2=(x+y)^4$.

13. $\left\{\left(\frac{a}{b}\right)^m-\left(\frac{b}{a}\right)^n\right\}^2$.

14. $(a-b+c)^2$.

15. $(a^2-b^2-c)^2$.

16. $(a-b)^2+(b-c)^2+(c-a)^2+2(a-b)(b-c)$
$-2(a-b)(c-a)-2(b-c)(c-a)$
$=\{a-b+b-c-(c-a)\}^2=(2a-2c)^2$.

17. $(2a^2-3b+4c)^2$.

Solve last three as in Example 4.

Exercise xxv.

1. $(7a+2b)(7a-2b)$.

2. $(3a+\frac{1}{2}b)(3a-\frac{1}{2}b)$.

3. $(9a^2-4b^2)(9a^2+4b^2)=(3a+2b)(3a-2b)(9a^2+4b^2)$.

4. $(10x-6y)(10x+6y)$.

5. $5b(a+2xy)(a-2xy)$.

6. $(3x^3-4y^2)(3x^3+4y^2)$.

7. $(\frac{3}{4}c+1)(\frac{3}{4}c-1)$.

8. $(2y^2-\frac{2}{3}xz)(2y^2+\frac{2}{3}xz)$.

9. $(9a^2-1)(9a^2+1)=(3a-1)(3a+1)(9a^2+1)$.

10. $(a^2-4b^2)(a^2+4b^2)(a-2b)(a+2b)(a^2+4b^2)$.

11. $a^{16}-b^{16}=(a^8-b^8)(a^8+b^8)=$
$(a^4-b^4)(a^4+b^4)(a^8+b^8)=$ &c.

12. $a^2-(b-c)^2=(a+b-c)(a-b+c)$.

13. $(a+2b+3x-4y)(a+2b-3x+4y)$.

14. $(x^2+2xy+y^2)(x^2-2xy+y)^2=(x+y)^2(x-y)^2$ or $(x^2-y^2)^2$

15. $(x+y+2z)(x+y-2z)$.

16. $(8x+8)(2-2x)=16(x+1))1-x)$.

17. $(2xy+x^2+y^2-z^2)(2xy-x^2-y^2+z^2)=$
$\{(x+y)^2-z^2\}\{z^2-(x-y)^2\}=$
$(x+y+z)(x+y-z)(z-x+y)(z+x-y)$

18. $(x^2+xy-y^2+x^2-xy-y^2)\times$
$\{x^2+xy+y^2-(x^2-xy+y^2)\}=(2x^2-2y^2)\times 2xy$
$=4xy(x+y)(x-y)$.

19. $(x^2-y^2+z^2-2xz)(x^2-y^2+z^2+xz)$
$=\{(x-z)^2-y^2\}\{(x+z)^2-y^2\}$
$=(x-z+y)(x-z-y)(x+z+y)(x+z-y)$.

20. $2(a+c)\times 2(b+d)=4(a+c)(b+d)$.

21. $4(1+2x^2)\times 6x=24x(1+2x^2)$.

22. $2(a^2+2ab+b^2)\times 4ab=8ab(a+b)^2$.

23. $(a^2-b^2+c^2-d^2+2ac-2bd)(a^2-b^2+c^2-d^2-2ac+2bd)$
$=\{(a+c)^2-(b+d)^2\}\{(a-c)^2-(b-d)^2\}=$
$(a+b+c+d)(a+c-b-d)(a-b-c+d)$
$\times(a+b-c-d)$.

24. $(x^2-y^2-z^2-2yz)(x^2-y^2-z^2+2yz)$
$=\{x^2-(y+z)^2\}\{x^2-(y-z)^2\}$
$=(x+y+z)(x-y-z)(x+y-z)(x-y+z)$.

25. $2(a^6-3a^3b^3+b^6)\times 4a^3b^3=8a^3b^3(a^6-3a^3b^3+b^6)$.

26. $(a^6+b^6)(a^6-b^6)+6a^3b^3(a^6-b^6)-8a^3b^3(a^6-b^6)$
$=(a^6-b^6)(a^6-2a^3b^3+b^6$
$=(a^3-b^3)(a^3+b^3)(a^3-b^3)^2=(a^3+b^3(a^3-b^3)^3.$

27. $(x^2+y^2+z^2)(x^2+y^2+z^2-2xy-2yz-2zx).$

28. $(x-y+z)^2-(y+z)^2=(x+2z)(x-2y).$

29. $-(a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2)$
$=-(a^2-b^2-c^2)^2+4b^2c^2$
$=(2bc+a^2-b^2-c^2)(2bc-a^2+b^2+c^2)$
$=\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}$
$=(a+b-c)(a-b+c)(b+c+a)(b+c-a).$

30. $x^4+y^4+z^4+2x^2y^2-2x^2z^2-2y^2z^2$
$=(x^2-y^2-z^2)-4y^2z^2$
$=\{x^2-(y^2-2yz+z^2)\}\{x^2-(y^2+2yz+z^2\}$
$=(x-y+z)(x+y-z)(x+y+z)(x-y-z).$

Exercise xxvi., page 67.

1. $(x-7)(x+2)$, $(x-7)(x-2)$; $(x+4)(x+3)$.
2. $(x-3)(x-5)$; $(x-7)(x-12)$; $(x-12)(x+5)$.
3. $(2x-5)(2x+4)=2(2x-5)(x+2)$;
$(3x-20)(3x-30)=3(3x-20)(x-10).$
4. $(\frac{1}{2}x+12)(\frac{1}{2}x-3)$;
$(5x+5)(5x+3)=5(x+1)(5x+3)$;
$(3x^3-4)(3x^3-5).$
5. $(\frac{1}{4}x+4)(\frac{1}{4}x+3)$;
$(4x^2-5)(4x^2+4)=4(4x-5)(x+1).$
6. $(x^2-a^2)(x^2-b^2)=$
$(x-a)(x+a)(x-b)(x+b)$;
$\{2(x+y)-11\}\{2(x+y)+9\}$
7. $(x^2+y^2-a^2)(x^2+y^2+b^2).$
8. $(a+b-3c)(a+b+c).$ 9. $\{x+y+(x+y)^2\}\{x+y+(x-y)^2\}$
$=(x+y)(1+x+y)\{x+y+(x-y)^2\}.$

10. $\{a+b-(a+b)^2\}\{a+b+(a-b)^2\}=$
$(a+b)(1-a-b)\{a+b+(a-b)^2\}$.

11. Last three terms $=-(2x+y)(x+2y)$ $\therefore$ the required factors are $(x^2+xy+y^2+2x+y)\times\{x^2+xy+y^2-(x+2y)\}$

12. $\{a-3(b-c)\}\{a+(b-c\}=(a-3b+3c)(a+b-c)$.

13. $[(x^3+y^3)^2+2a^3(x^3+y^3)+a^6]-b^6=$
$(x^3+y^3+a^3)^2-b^6=$&c.

14. $(x^2-10x-12)(x^2-10x+8)$.

15. $(x^2-14x+40-30)(x^2-14x+40+5)$
$=(x^2-14x+10)(x^2-14x+45)$
$=(x^2-14x+10)(x-9)(x-5)$.

16. $\{(x^2-xy+y^2)+3xy\}(x^2-xy+y^2-xy)$
$=(x+y)^2(x-y)^2=(x^2-y^2)^2$.

17. $(z^2-1)(z^2-2)=(z+1)(z-1)(z^2-2)$;
$(x^2-3)(x^2+1)$; $(3x^4+5y^2)(3x^4-2y^2)$.

18. $(c^m+2)(c^m-1)$; $(x^3-2)(x^3+1)$
$=(x^3-2)(x+1)(x^2-x+1)$; $(x^m-4y^n)(x^m+2y^n)$.

19. $(x^m-ay^n)(x^m+by^m)$.

Exercise xxvii.

1. $(x-by)(bx-y)$.
2. $(3x+by)(2x-y)=3(x+2y)(2x-y)$.
3. $(56x-20y)(x-y)=4(14x-5y)(x-y)$
4. $(56x-20y)(x-y)=4(14x-5y)(x-y)$.
5. $(56x-4)(x-20y)$.
6. $(28x-20y)(2x-y)=4(7x-5y)(2x-y)$.
7. $(28x+y)(2x-20y)=2(28x+y)(x-10y)$.
8. $(28x-10y)(2x+2y)=4(14x-5y)(x+y)$.
9. $(8x-5y)(7x-4y)$.
10. $(8x+5y(7x-4y)$.
11. $(3x+y)(2x-by)=2(3x+y)(x-3y)$
12. $(3x-2y)(2x+3y)$.
13. $(28x+y)(2x+20y)=2(28x+y)(x+10y)$.

14. $(56x-10y)(x-2y)=2(28x-5y)(x-2y)$.

15. $(28x+5y)(2x-4y)=2(28x+5y)(x-2y)$.

16. $(56x-5y)(x-4y)$.

17. $(8x-2y)(7x-10y)=2(4x-y)(7x-10y)$.

18. $(56x+4y)(x-5y)=4(14x+y)(x-5y)$.

19. $(9x+3y)(4x-5y)=3(3x+y)(4x-5y)$.

20. $(8x+5y)(9x-8y)$.

Exercise xxviii.

1. $(5x-7)(2x+3)$.

2. $(5x+3)(2x-7)$.

3. $(5x-3)(2x+7)$.

4. $(2x-5)(3x-11)$.

5. $(4a+1)(3a-2)$.

6. $(3x-7)(4x-3)$.

7. $(3x+7)(4x+3)$.

8. $(5a^3-4b^2)(3a^2+5b^3)$.

8. $(5a^3-4b^2)(3a^2+5b^2)$.

9. $(4x+1)(3x-1)$.

10. $3y^2(x-y)(3x+2y)$.

11. $(2x+3y)(2x+y)$.

12. $x^2(3b+x)(2b-3x)$.

13. $(3x^2+7y^2)(2x^2-5y^2)$.

14. $(2x^2-9)(x^2+5)$.

15. $(4x^2-y^2)(x^2-9y^2)$
$=(2x+y)(2x-y)(x-3y)(x+3y)$.

16. Here we have exactly the 15th with $x+2$ for x. $\therefore$ the factors will be found by changing x into $x+2$ in the factors of the 15th.

$$(2x+4+y)(2x+4-y)(x+2-3y)(x+2+3y).$$

17. Let $z=(2x+3y)$ and $a=(3x-2y)$.

It becomes $6z^2+5az-6a^2=(3z-2a)(2z+3a)$.

$\therefore$ Ans. $(6x+9y-6x+4y)(4x+6y+9x-6y)$
$=13y\times 13x=169xy$.

18. Making same substitutions as in the 17th,

$6z^4+5z^2a^2-6a^4=(3z^2-2a^2)(2z^2+3a^2)$
$=(12x^2+36xy+27y^2-18x^2+24xy-8y^2)\times$
$(8x^2+24xy+18y^2+27x^2-36xy+12y^2)$
$=(19y^2+60xy-6x^2)(35x^2-12xy+30y^2)$.

19. Let $a=(x^2+y^2+xy)$, $b=(x^2-xy+y^2)$,

$\therefore\ 6a^2+13ab-385b^2=(a-7b)(6a+55b)$

$=(8xy-6x^2-6y^2)(61x^2+49xy+61y^2)$

$=2(4xy-3x^2-3y^2)(61x^2-49xy+61y^2)$.

20. Let $a=(x^2+2xy+2y^2)$, $b=(x^2-2xy+2y^2)$,

$\therefore\ (21a^2-5ab-6b^2)=(7a+3b)(3a-2b)$

$=(10x^2+8xy+20y^2)(x^2+10xy+2y^2)$

$=2(5x^2+4xy+10y^2)(x^2+10xy+2y^2)$.

Exercise xxix., page 73.

1. $(7x+6y+8)(x-y-z)$.
2. $(5x-5y-22)(4x+y+4)$.
3. $(3x^2+4y^2+13)(x^2-y^2-1.)$
4. $(4x+5y)(5x-4y+7)$.
5. $(9x+8y-20)(8x-y-1)$.
6. $(x+3y)(x-4y-5)$.
7. $(4x+3y-z)(2x+3y+z)$.
8. $(3x-2y-2z)(2x-3y+4z)$
9. $(3x^2-2y^2+5z^2)(2x^2+5y^2-5)$.
10. $(15x^2+8y^2+5z^2)(x^2-2y^2+3z^2)$.
11. $(2a-5b-7c)(2a+3b+3c)$.
12. Expression $=(a^2-b^2-c^2)^2-4b^2c^2$

$$=(a^2-b^2-c^2+2bc)(a^2-b^2-c^2-2bc)$$
$$=\{a^2-(b-c)^2\}\{a^2-(b+c)^2\}$$
$$=(a-b+c)(a+b-c)(a+b+c)(a-b-c).$$

Hints and Solutions.

Most of these examples can be factored by the method of Art. XVII; but a few solutions by the method of next article may be given.

1. $(7x+6y)(x-y)$; $(7x+8)(x-2)$; $(-y-z)(6y+8)$

$7x+6y$, $7x+8$, $6y+8$, $x-y$, $x-z$, $-y-z$. Hence the factors are, &c.

3. Factoring the partial products as in Art. XVIII, and arranging in two lines:

$$\begin{array}{lll} 3x^2+4y^2, & 3x^2+13, & 4y^2+13 \\ x^2-\ y^2, & x^2-\ 1, & -y^2-\ 1. \end{array}$$

Hence the factors are $3x^2+4y^2+13$ and x^2-y^2-1.

NOTE.—In cases where numerical terms appear, consider these terms as involving z^0.

5. $(9x+8y)(8x-y)$; $(9x-20)(8x-1)$; $(8y-20)(-y-1)$, and arranging in two lines as before.

$$\begin{array}{lll} 9x+8y, & 9x-20, & 8y-20 \\ 8x-\ y, & 8x-\ 1, & -y-\ 1 \end{array}$$

$\therefore$ the factors are, &c.

9. Omit terms involving z, $6x^4-10y^4+11x^2y^2+10y^2-15x^2$

Factoring the first three terms we have $3x^2-2y^2$
$2x^2+5y^2$; but the last two terms show that the third term of second factor must be -5, hence, &c.

Exercise xxx., page 75.

1. Quantity $=x^4+7x^2+\frac{49}{4}-\frac{45}{4}=(x^2+\frac{7}{2})^2-\frac{45}{4}$, which gives the factors $x^2+\frac{7}{2}\pm\frac{3}{2}\sqrt{5}$; $2x^2+\frac{7}{2}\pm\frac{3}{2}\sqrt{5}$.

2. $x^2+\frac{7}{2}y^2\pm\frac{3}{2}y^2\sqrt{5}$, as in last Ex. ;

quantity $=\frac{1}{12}(36x^4+60x^2y^2+25y^4-13y^4)$

$=\frac{1}{12}\{(6x^2+5y^2)^2-13y^4\}$, which gives the factors

$\frac{1}{12}(6x^2+5y^2\pm y^2\sqrt{13})$.

3. Quantity $=\frac{1}{4}(16x^4+40x^2+25-13)$, which gives $\frac{1}{4}(4x^2+5\pm\sqrt{13})$; writing k for $x+y$, the quantity $=3k^4+5k^2z^2+z^4$, which factor as in second part of Ex. 2 above, and restore the value of k and factors are

$\frac{1}{12}\{6(x+y)^2+5z^2\pm z^2\sqrt{13}\}$.

4. $(x^2+\frac{1}{2}y^2)(x^2+6\frac{1}{2}y^2)$; $(x^2+\frac{11}{2}y^2)(x^2+\frac{3}{2}y^2)$.

5. $(2x^2+4\frac{1}{4}y^2)(2x^2+\frac{1}{4}y^2)$; $\frac{1}{4}\{4(a+b)^2+3c\pm\sqrt{13}\}$.

6. Multiply by 4×3, quantity $=\frac{1}{12}(36x^4+96x^2y^2+55y^4)$

$=\frac{1}{12}(6x^2+5y^2)(6x^2+11y^2)$;

Second part derived from the first by putting $y=1$,

$\therefore$ factors are $(6x^2+5)(6x^2+11)$..

7. $\frac{1}{5}(5x^2+10\pm3\sqrt{10})$; $(2a^2+3\pm2\sqrt{2})$.

8. From second part of last Ex. we write at once

$\{2(x+y)^2+(3\pm2\sqrt{2})z^2\}$; from first part

$\frac{1}{5}\{10x^2+(10\pm3\sqrt{10})y^2\}\{10x^2+(20-6\sqrt{10})y^2\}$.

9. Quantity $=\frac{1}{9}(81x^4+126x^2+36)=\frac{1}{9}(81x^4+126x^2+49-13)$

$=\frac{1}{9}(9x^2+7\pm\sqrt{13})$.

In second put m for $y+z$, and afterwards restore the values; factors are

$$\tfrac{1}{2}\{2x^2+(6\pm\sqrt{16})(y+z)^2\}.$$

10. $\frac{1}{2}(2x^2+6\pm\sqrt{6})$. In second quantity

$=\frac{1}{7}(49x^4+280x^2+400-85)$, which gives $\frac{1}{7}(7x^2+20\pm\sqrt{85})$.

11. Quantity $=\frac{1}{2}(16x^4+72x^2y^2+58y^4)$

$=\frac{1}{2}(16x^4+72x^2y^2+81y^4-23y^4)$, which gives

$\frac{1}{2}\{4x^2+(9\pm\sqrt{23})y^2\}$

12. $\frac{1}{7}\{7(a-b)^2+8c^2\pm c\sqrt{29}\}$; $\frac{1}{6}\{3a^2\pm b^2\sqrt{3}\}$.

13. $\frac{1}{3}\{3x^2+(3\pm\sqrt{3})y^2\}$; in this write $a+b$ for x, $a-b$ for y

$\therefore$ $\frac{1}{3}\{3(a+b)^2+(3\pm\sqrt{3})(a-b)^2\}$.

14. $\{7a^2+(6\pm\sqrt{14})b^2\}$; $(5m^2+9n^2)(5m^2+3n^2)$.

15. $\{7(m+n)^2+(6\pm\sqrt{14})(m-n)^2\}$

Exercise xxxi., page 77.

1. $(x^2\pm2xy+3y^2$; $(x^2\pm xy-y^2)$; $x^2\pm xy+y^2)$.

2. $(x^2\pm2xy+2y^2$; $(4x^2\pm3xy+y^2)$; $(\frac{1}{2}x^2\pm xy+y^2)$.

3. $(x^2\pm\sqrt{2}x+1)$; $(x^2\pm\sqrt{6}xy+3y^2)$; $1\pm2y-4y^2)$.

4. $(x^2\pm3x+1)$; $(x^2\pm\sqrt{6}x+3)$; $(\frac{1}{2}x^2\pm2xy+y^2$.

5. Expn. $=y^4+11x^2y^2+\frac{121}{4}x^4-\frac{125}{4}x^4=y^2+\frac{11}{2}x^2\pm\frac{5}{2}x^2\sqrt{5}$;

Expn. $=x^8+4x^4y^4+y^8-4x^4y^4$, which gives the factors

$$x^4+2y^4\pm2x^2y^2;$$

Expn. $=x^4+8x^2+16-4x^2=$ the factors $x^2+4\pm2x$.

6. Expn. $=(2x^2+y^2)^2-12\frac{1}{4}x^2y^2=2x^2+y^2\pm\frac{7}{2}xy$;

$(x^2+y^2)^2-\frac{39}{16}x^2y^2=x^2+y^2\pm\frac{1}{4}xy\sqrt{39}$;

$4x^4+1=(2x^2+1)^2-4x^2=2x^2+1\pm 2x$.

7. $x^{2m}+8y^{2m}\pm 4x^my^m$; $x^{2m}+2y^{2m}\pm 2x^my^m$; $\frac{1}{2}x^2-\frac{3}{4}y^2\pm xy\sqrt{5}$.

8. $2x^2-1\pm 2x$; Expn. $=-(\frac{1}{4}x^4-6x^2y^2+36y^4-x^2y^2_{\infty})$

$=-(\frac{1}{2}x^2-6y^2+xy)(\frac{1}{2}x^2-6y^2-xy)$;

Expn. $=x^4+2a^2x^2y^2+a^4y^4-2a^2x^2y^2=x^2+a^2y^2\pm axy\sqrt{2}$

9. $mx^2-ny^2\pm xy\sqrt{p}$; expn. $=x^{4m}+2\,.\,2^{m-1}y^{2m}-2.2^{2m-1}y^{2m}$

$=x^{2m}+2^{m-1}y^{2m}\pm 2^my^m$.

10. Expn. $=(4x^2-3)^2-x^2=4x^2-3\pm x$; $2x^2-2\pm 2x\sqrt{2}$;

Expn. $=+(9x^4-12x^2y^2+y^4)=$

$-(3x^2-2y^2+xy)(3x^2-2y^2-xy)$.

11. $(2x^2\pm\frac{4}{5}xy-3y^2)$; $x^2\pm 2x+5)$.

12. $a^4+b^4+(a+b)^4=(a^2+b^2)^2-a^2b^2+(a+b)^4-a^2b^2$

$=(a^2+ab+b^2)(a^2-ab+b^2)+(a^2+3ab+b^2)(a^2+ab+b^2)$

$=(a^2+ab+b^2)(2a^2+2ab+2b^2)$

$=2(a^2+ab+b^2)(a^2+ab+b^2)=2(a^2+ab+b^2)^2$.

If $b=1$ we get the second part $2(a^2+a+1)^2$.

13. $\{(x+y)^2+3(x+y)z+z^2\}\{(x+y)^2-3(x+y)z+z^2\}$.

14. Expn. $=(a+b)^4+7c^2(a+b)^2+\frac{49}{4}c^4-\frac{45}{4}c^4$

$=(a+b)^2+\frac{7}{2}c^2\pm\frac{3}{2}c^2\sqrt{5}$.

15. $\{4a^2+5a(b-c)+2(b-c)^2\}\{4a^2-5a(b-c)+2(b-c)^2\}$

16. $\{2(a+b)^2\pm 3(a^2-b^2)-3(a-b)^2\}=$

$4(a^2+5ab-2b^2)(b^2+5ab-2a^2)$.

17. From 13th,

$\{(x^2+y^2-xy)^2\pm 3(x^2+y^2-xy)(x+y)+(x+y)^2\}$

18. From 14th, $\{a^2+ab+b^2)+\frac{7}{2}(a-b)^2\pm\frac{3}{2}(a-b)^2\sqrt{5}\}$.

19. $4a^2\pm 2a+1)$; $(x^2\pm 7x+4)$.

20. $(x^2\pm 9xy+9y^2)$; $(1\pm 3z+5z^2)$.

21. $\{(a^2+1)^2 \pm 2(a^2+1)a + 4a^2\} = (3a^4+8a^2+1)(1+4a^2-a^4)$

$\{(x+1)^2 \pm 2(x^2-1)+3(x-1)^2\}$

$=(6x^2-4x+2)(2x^2-4x+6)=4(3x^2-2x+1)(x^2-2x+3).$

Exercise xxxii., page 78.

1. $(x^2+3)(x^2+2x-3)=(x^2+3)(x+3)(x-1).$

2. $2(x^2+3)(x^2+x-3).$

3. $(x^2+4)(x^2+3x-4)=(x^2+4)(x+4)(x-1).$

4. $(x^2-4)(3x^2+x+12)=(x+2)(x-2)(3x^2+x+12).$

5. $(x^2-3)(5x^2+4x+15).$ 6. $(x^2+6)(10x^2+5x-60).$

7. $(\frac{1}{2}x^2+\frac{1}{10})(\frac{1}{2}x^2+40x-\frac{1}{10}).$ 8. $(5x^2-1)(5x^2-8x+1).$

9. $(5x^2-8)(7\frac{1}{2}x^2-6x-12).$

Expression $=1\frac{1}{2}(25x^4-64)-6x(5x-8).$

10. $7(9x^4-16)-13x(3x^2-4)=(3x^2-4)(21x^2-13x-28).$

11. $10(81x^4-\frac{1}{4})+\frac{9}{4}x(\frac{9}{4}x^2+\frac{1}{2})=(18x^2+1)(45x^2+\frac{9}{8}x+\frac{5}{2}).$

12. $2(121x^4-1)-3x(11x^2+1)=(11x^2+1)(22x^2-3x-2).$

13. $(\frac{1}{2}x^2-\frac{2}{5})(\frac{1}{2}x^2+\frac{1}{5}x+\frac{2}{5}).$

14. $8(x^2-2y^2)(10x^2-4xy+20y^2).$

15. $(2x^2-5y^2)(12x^2-6xy+30y^2).$

16. $(x^2-16y^2)\{2x^2+\frac{1}{2}xy+32y^2)$

17. $(x^2-\frac{6}{5})(11x^2+10x+\frac{66}{5}).$

18. $10(x^2+2)(4x^2+3x-8).$

19. $(x^2-6y^2)(13x^2-12xy+78y^2).$

20. $(x^2+4y^2)(3x^2+3xy-12y^2).$

21. $(x^2-3y^2)(5x^2+4xy+15y^2).$

22. $2(x^2-2y^2)(2x^2-7xy+2y^2).$

23. $(x^2+\frac{1}{5}y^2)(x^2+80xy-\frac{1}{5}y^2).$

24. $(x^2-6y^2)(2x^2-xy+12y^2.)$

Exercise xxxiii., page 79.

1. $(x^2+27)^2-6x(x^2+27)-27x^2=$
$(x^2+27)^2-6x(x^2+27)+9x^2-36x^2$
$=(x^2+3x+27)(x^2-9x+27).$

2. $x^4+8x^2-5x^2+2x(x^2+4)=\{(x^2+4)^2+2x(x^2+4)+x^2\}$
$-6x^2=x^2+x(1\pm\sqrt{3})+4.$

3. $\{x^2+1+\frac{1}{2}(1\pm\sqrt{5})x\}.$

4. $(x^2+1)^2-4x(x^2+1)+4x^2-5x^2=x^2+1-x(2\pm\sqrt{5}).$

5. $\{2x^2+2-3x\pm x\sqrt{(23)}\}$

6. $(x^2+15x-5)(x^2-x-5).$

7. $(4x^2-2)(4x^2-6x-2).$

8. $(x^2+8x+4)(x^2-3x+4).$

9. $(x^2+7x-2)(x^2-x-2).$

10 $(x^2+5xy+3y^2)(x^2-xy+3y^2)$

11. $(x^2-1)^2+6x(x^2-1)+9x^2-16x^2=$
$(x^2+10x-1)(x^2+2x-1).$

12. $(x^2+7xy+y^2)(x^2-3xy+y^2).$

13. $(2x^2-5y^2)^2+2xy(x^2-5y^2)+x^2y^2-46x^2y^2=$
$(2x^2+xy-5y^2\pm xy\sqrt{46}.)$

14. $(x^2+7xy-y^2)(x^2-xy-y^2).$

15. $(x^2+2y^2)^2+6xy(x^2+2y^2)+9x^2y^2-3x^2y^2=$
$(x^2+2y^2+3xy\pm xy\sqrt{3}).$

16. $(3x^2+10xy-2y^2)(3x^2-4xy-2y^2).$

17. $\frac{1}{11}\{121x^4+484x^2y^2+484y^4+10xy(11x^2+22y^2)$
$-46\frac{3}{11}x^2y^2\}=\frac{1}{11}\{11x^2+22y^2+5xy\pm\frac{28}{11}xy\sqrt{11}\}.$

Exercise xxxiv., page 78.

1. $(y-z)(x^2-y).$

2. $ax(by+c)+(by-c)(by+c)=(by+c)(ax+by-c).$

3. $(z^2+a)(x+a)(x-a).$

4. $(2x-a)(x-2b).$

5. $(x+3a)(x+2b).$

6. $(x-b^2)(x-a)(x+a).$

7. $(x-b)(x+b)(x-a)(x^2+ax+a^2).$

8. $(2x+3a)(4x+5b)$. 9. $(a+bx)(a-bx+cx^2)$.

10. $(a+bx)(a-bx)+cx^2(a+bx)=(a-bx)(a+bx+cx^2)$.

11. Group the terms containing a, also those containing b.

$\therefore\ (ax-d)(bx^2+cx-f)$.

12. $(px-q)(x^2-x-1)$.

13. If written $a^2+ab+2ac-2b^2+5bc-3c^2$ it becomes

$(a-b-c)(a+2b+3c)$.

14. $x^3+x^2+x+ax^2+ax+a=(x+a)(x^2+x+1)$.

15. $(mx-n)(px^2+qx-r)$.

16. $x(x^2-bx-ax+bc)-a(x^2-bx-ax+bc)=$

$(x-a)(x-b)(x-c)$

17. $x^3-(b+c)x^2+bcx+ax^2-a(b+c)x+abc$

$=x(x^2-\overline{b+c}x+bc)+a(x^2+\overline{b+c}x+bc)=(x+a)(x-b)(x-c)$

Group terms as in last example.

18. $\therefore\ (x+a)(x+b)(x-c)$.

19. $a^2(x^3-ax^2y-xy+ay^2)+z(x^3-ax^2y-xy+ay^2)$

$=(a^2+z)(x^3-ax^2y-xy+ay^2)=(a^2+z)(x-ay)(x^2-y)$.

20. $abx(ax+by)+cdy(ax+by)-efz(ax+by)\cdot$

$(abx+cdy-efz)(ax+by)$.

21. $ax(ax^2-bx+c)+c(ax^2-bx+c)=(ax+c)(ax^2-bx+c)$.

22. $mx(x^2-y^2)-ny(x^2-y^2)+rz(x^2-y^2)$

$=(x^2-y^2)(mx-ny+rz)=(x-y)(x+y)(mx-ny+rz)$.

23. $ax(mx-ny)+by(mx-ny)+cz(mx-ny)$

$(mx-ny)(ax+by+cz)$

24. $mx(ax-bcx+a)+n(ax-bcx+a)=(mx+n)(ax-bcx+a)$.

25. $c^2(a^2b^2-b^2xy-a^2yz+xy^2z)$

$-xz(a^2b^2-b^2xy-a^2yz+xy^2z)=(c^2-xz)(b^2-yz)(a^2-xy)$.

26. Arrange in three groups, terms in m^2, terms in a, and remaining ones.

$\therefore\ x^3\{x^2-(n-n^2)\}-m^2x^2\{x^2-\overline{n-n^2}\}-a(x^2-\overline{n-n^2})$

$=(x^3-m^2x^2-a)(x^2-\overline{n-n^2})$.

27. $(1+x-x^2)-ax(1+x-x^2)+bx^2(1+x-x^2)$
$-cx^3(1+x-x^2)=(1+x-x^2)(1-ax+bx^2-cx^3)$.

28. Group terms in d, $\therefore ax(a^2x^2-abxy+acxy-bcy^2)$
$-dy(a^2x^2-abxy+acxy-bcy^2)=(ax-dy)(ax-by)(ax+cy)$.

29. $mx(m^2px^2-npx+m^2nx-n^2)$
$+q(m^2px^2-npx+m^2nx-n^2)=(mx+q)(px+n)(m^2x-n)$.

30. $m^2x^2(p^2x^3+p^2x^2+q^2xy^2+q^2y^2)-$
$n^2y^2(p^2x^3+p^2x^2+q^2xy+q^2y^2)=$
$(mx+ny)(mx-ny)(p^2x^2+q^2y^2)(x+1)$.

Exercise xxxv., page 82.

1. $(a+x)(a-b)$. 2. $(ax+by)(bx-ay)$.

3. $x^4-a^4+ax(x^2-a^2)=(x-a)(x+a)(x^2+ax+a^2)$.

4. $x(a+x)(a^2+ax+x^2)$. 5. $(ax-b)(cx+d)$.

6. $25x^4-5x^3+x^2-1=$
$(25x^4-1)-x(5x^2-1)=(5x^2-1)(5x^2-x+1)$.

7. $(a-b)(a+b+x-c)$. 8. $(a^2+b)(a+b)$.

9. $(x^2-y^2)(x^2+2xy+y^2)=(x-y)(x+y)^3$.

10. $(x-y+1)(x^2+xy+y^2)$. 11. $(b-2x)(2+bx)$.

12. $x^3-1+3(x^2-1)=(x-1)(x^2+4x+4)=(x-1)(x+2)^2$.

13. $(p-q)(p^2-2q^2)$.

14. $a^3+a^2-2=a^3-a^2+2a^2-2=a^2(a-1)+2(a^2-1)$
$=(a-1)(a^2+2a+2)$.

15. $3a^2b^4-2ab^2-1=3a^2b^4-3ab^2+ab^2-1$
$=3ab^2(ab^2-1)+(ab^2-1)=(ab^2-1)(3ab^2+1)$.

16. $y(y^2-1)-2(y-1)=(y-1)(y^2+y-2)=(y-1)^2(y+2)$.

17. $2a^3-a^2b-ab^2+2b^3=2(a^3+b^3)-ab(a+b)$
$=(a+b)(2a^2-3ab+2b^2)$.

18. $b^{3m}-1+b^{2m}-1=(b^m-1)(b^{2m}+2b^m+2)$.

19. $y^{3n}-2y^{2n}z^n-2y^nz^{2n}+z^{3n}=(y^n+z^n)(y^{2n}-3y^nz^n+z^{2n})$.

20. $a(a^2-b^2)-3b^2(a-b)=(a-b)(a^2+ab-3b^2)$.

21. $a^m(a^m-c^n)-2c^n(a^m-c^n)=(a^m-c^n)(a^m-2c^n)$.

22. $(ax-b)(x^2-ax-b)$.

23. $35x^{2n}+15a^2x^n-21a^2x^n-9a^4=5b^n(7x^n+3a^2)-3a^2$
$(7x^n+3a^2)=(5x^n-3a^2)(7x^n+3a^2)$.

24. $a^2b^2-(bc-ca)^2=(ab+bc-ca)(ab-bc+ca)$.

25. $(m^2-b^2)(a-m)=(m-b)(m+b)(a-m)$.

26. $(\frac{1}{3}-3a^2)(1-9a^2)=(\frac{1}{3}-3a^2)(1-3a)(1+3a)$.

27. $(x-y)^3-(x-y)^2z+(x-y)z-z^2$
$=(x-y)^2(x-y-z)+z(x-y-y)$
$=(x-y-z)(x^2-2xy+y^2+z)$

28. $6m(4m^2+n^2)-7n(4m^2+n^2)=(6m-7n)(4m^2+n^2)$.

29. $x^m(x^n+y^m)+y^n(x^n+y^m)=(x^m+y^n)(x^n+y^m)$.

30. $x^4+2x^3y+x^2y^2-(a^2x^2+2axy^2+y^4)$
$=(x^2+xy)^2-(ax+y^2)^2$
$=(x^2+xy+ax+y^2)(x^2+xy-ax-y^2)$.

Exercise xxxvi., page 84.

1. $(x^3-y^3)(x^3+y^3)$
$=(x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$;
$(x-1)(x^2+x+1)$; $(x+2)(x^2-2x+4)$;
$(2a-3x)(4a^2+6ax+9x^2)$; $(2+ax)(4-2ax+a^2x^2)$.

2. $x^5-(a^2)^5=(x-a^2)(x^4+x^3a^2+x^2a^4+xa^6+a^8)$;
$(3a-4)(9a^2+12a+16)$; $(a^6-b^4)(a^6+b^4)$;
$=(a^3-b^2)(a^3+b^2)(a^6+b^4)$; $(x^2)^5-(2y)^5=$
$(x^2-2y)(x^8+2x^6y+4x^4y^2+8x^2y^3+16y^4)$.

3. $(a-b)$. 4. $(x+4y)$.

5. Expn. $=\dfrac{x^8-y^8}{x-y} = \dfrac{(x-y)(x+y)(x^2+y^2)(x^4+y^4)}{x-y}$
$=(x+y)(x^2+y^2)(x^4+y^4)$.

6. $\{(3y^2-2x^2)-(3x^2-2y^2)\}\{(3y^2-2x^2)^2+(3y^2-2x^2)\times$
$(3x^2-2y^2)+(3x^2-2y^2)^2\}$
$=5(y^2-x^2)(7x^4-11x^2y^2+7y^4)$;
$(a^4-4b^2)(a^4+4b^2)=(a^2-2b)(a^2+2b)(a^4+4b^2)$.

7. $(x-y)(x^2+xy+y^2)-x(x+y)(x-y)+y(x-y)$
$=(x-y)(x^2+xy+y^2-x^2-xy+y)=y(x-y)(y+1)$.

8. $b(x^3-a^3)+ax(x^2-a^2)+a^3(x-a)$
$=(x-a)\{b(x^2+ax+a^2)+ax(x+a)+a^3\}$
$=(x-a)\{(a+b)x^2+ax(a+b)+a^2(a+b)\}$
$=(x-a)(x^2+ax+a^2)(a+b)$.

9. $b(m^3+a^3)+am(m^2-a^2)+a^3(m+a)$
$=b(m^3+a^3)+a(m^3+a^3)=(a+b)(m+a)(m^2-am+a^2)$.

10. $(x^2-y^2)(x^4+x^2y^2+y^4)+2xy(x^2+x^2y^2+y^4)$
$=(x^4+x^2y^2+y^4)(x^2+2xy-y^2)$
$=(x^2+xy+y^2)(x^2-xy+y^2)(x^2+2xy-y^2)$.

11. $(a^2-bc+2bc)\{(a^2-bc)^2-2bc(a^2-bc)+4b^2c^2\}=$
$(a^2+bc)(a^4-4a^2bc+7b^2c^2)$.

12. $(x-a)^3+b^3=(x-a+b)\{(x-a)^2-(x-a)b+b^2\}$.

13. $(x+2y)(x^2-2xy+4y^2)+4xy(x^2-2xy+4y^2)$
$=(x^2-2xy+4y^2)(x+2y+4xy)$.

14. $8x^3+27y^3-6xy(2x+3y)=(2x+3y)(4x^2-6xy+9y^2-6xy)$
$=(2x+3y)(2x-3y)^2$.

15. $1-2x+4x^2(1-2x)=(1-2x)(1+4x^2)$.

16. The expression

$$=\frac{a^6-b^6c^6}{a-bc}=\frac{(a^3-b^3c^3)(a^3+b^3c^3)}{a-bc}$$
$$=(a^2+abc+b^2c^2)(a+bc)(a^2-abc+b^2c^2).$$

Exercise xxxvii., page 89.

1. Putting $x+y=0$ the expression vanishes.

$\therefore\ (x+y+z)^3-(x^3+y^3+z^3)=m(x+y)(y+z)(z+x)$.

Put $x=1,\ y=1,\ z=1$,

$\therefore\ 24=8m\ \therefore\ m=3$.

$\therefore$ Expression $=3(x+y)(y+z)(z+x)$.

2. $bc(b-c)+ca(c-a)+ab(a-b)=(a-b)(b-c)(a-c)$.

3. Proceed as in 1 above, $3\,(a^2-b^2)(b^2-c^2)(c^2-a^2)$.

4. $(x+y)(y+z)(z+x)$.

5. $3(a+b)(b+c)(c+a)$.

6. See Hand-book, Ex. 5. $(a+b+c)(a-b)(b-c)(c-a)$.

7. $(a+b)(b+c)(c+a)$.

8. Put $c-b^2=0$, $\therefore\ a^3(c-b^2)+b^3(a-b^4)+b^6(b-a^2)$
$+ab^3(ab^3-1)=ab^3-b^7+b^7-a^2b^6+a^2b^2-ab^3=0$

$\therefore\ c-b^2$ is a factor, &c., result is $(a^2-b)(b^2-c)(c^2-a)$.

9. $(a+b)(b+c)(c+a)$.

10. Expn. is of three dimensions and vanishes when $a-b=0$, $\therefore\ (a-b)(b-c)(c-a)$.

11. $(x^2+y^2(y^2+z^2)(z^2+x^2)$.

12. $a-b$ is a factor, $\therefore\ b-c,\ c-a$ are factors. So that $(a-b)(b-c)(c-a)$ is a factor: the remaining factor is of *two* dimensions and *symmetrical* in a, b, c, and must $\therefore$ be of the form $m(a^2+b^2+c^2)+n(ab+bc+ca)$, where m, n are numerical, and may be positive, or negative, or zero; to determine them put $c=0$, $\therefore\ (a-b)^5+b^5-a^5=-ab(a-b)\{m(a^2+b^2+nab\}$ multiplied by some numerical quantity, p, suppose, (independent of the letters) to be afterwards determined. Hence

$-5ab(a-b)\{a^2+ab+b^2-2ab\}=-pab(a-b)\{m(a^2+b^2)+nab\}$
whence $a^2+b^2-ab=m(a^2+b^2)+nab$, $\therefore\ m=1,\ n=-1$:

$\therefore$ the quad. factor required is $a^2+b^2+c^2-ab-bc-ca$.

13. Put $a+b+c=0$, *i.e.*, $a+b=-c$, &c., $\therefore$ $-abc-abc-abc+(a+b)^3+c^3-3ab(a+b)=-3abc-c^3+3abc+c^3=0$. As in last Example, the other factor is a symmetric quadratic in a, b, c, and $\therefore$ of the form $m(a^2+b^2+c^2)+n(ab+bc+ca)$.

Put $c=0$, $\therefore$ $ab(a+b)+(a^3+b^3)=(a+b+0)\{m(a^2+b^2+0^2)+n(ab+0+0)\}$ multiplied by coefficient which is independent of m, n, and to be afterwards determined: hence $ab+(a^2-ab+b^2)=m(a^2+b^2)+nab$, $\therefore$ $m=1$, $n=0$, and required factor is $a^2+b^2+c^2$.

14. $(c-b^3)(a-c^3)(b-a^3)$. 15. $(x^2-y^2)(y^2-z^2)(x^2-z^2)$.

16. $(x+y+z)(x-y+z)(y-z+x)(z+y-x)$.

17. $(a-b)(b-c)(a-c)$.

18. By formula [8] expn. $=\{2(a+b+c)\}^3=8(a+b+c)^3$;

Or substituting $a+b+c=0$ we get the same.

19. Substituting a^2 for b the expression vanishes $\therefore$ a^2-b is a factor.

20. $(x+y)^7-x^7-y^7=7xy(x+y)(x^4+2x^3y+3x^2y^2+2xy^3+y^4)=7xy(x+y)(x^2+xy+y^2)^2$.

21. Substitute $x^2=5x-6$, $5x^2-6x-45x+54+26x-24$.

Substituting again $25x-30-6x-45x+54+26x-24=0$.

$\therefore$ x^2-5x+6 is a factor.

22. Substitute $a=b-c$ $\therefore$ $(b-c)^2(b+c)-b^3+c^2(2b-c)+bc(b-c)=(b^2-c^2)(b-c)-b^3+c^2(2b-c)+bc(b-c)=0$.

$\therefore$ $a-b+c$ is a factor.

23. Substituting $a^3=z-3b$

$9b^2+12ab^4-9b^5-9b^2-12ab^4+9b^5=0$, $\therefore$ a^3+3b is a factor.

24. $(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+bc+ca)$.

Note.—The quadratic factor found as in 13 above.

Exercise xxxviii., page 96.

1. $(a-2)(a^2-7a+2)$. 2. $(x-2)(x-3)(x-4)$.

3. $(x-3)(x-2)^2$. 4. $(x-2)^2(x+4)$.

5. $(x+1)(x^2+2x+3)$. 6. $(x^2+2x+3)(x^2+2x+3)$.

7. $(x+2)(x-1)^2$.
8. $(x^2+2x+3)(x^2-2x+3)$.
9. $(m-n)(m^2-2mn-2n^2)$.
10. None.
11. $(m-n)(m-2n)^2$.
12. $(b+3c)(b^2-2bc+13b^2)$.
13. $-(m-n)^2(m^2-mn+n)$.
14. $(a+2b)(a-2b)(a^2-7ab+4b^2)$.
15. $(x-5)(x-3)^2$.
16. $(x+2)(x^2+3x+1)$.
17. $(a-1)(a^2-2a-195)$.
18. $(p+2)(p-1)(p+4)$.
19. $(a-1)^2(a+2)(a+3)$.
20. $(a^{2n}-1)(a^{2n}-2)(a^{2n}-3)$.
21. $a^2+4b^2\pm 7ab$.
22. $(a-b)^2(a^2+2ab+2b^2)$.
23. $(p-2)(p^2-2p+2)$.
24. $(x^n-1)(x^{2n}+5x^n+5)$.
25. $(y-2)(y^3-3y^2+2y+4)$.
26. None.
27. $(a-b)(a^2+2ab+3b^2)$.
28. $(a^n+1)(2a^{2n}-3a^n+2)$.
29. $(x-2)(x-3)(x-6)(x-7)$.
30. $(x-y)(x-2y)(x-3y)^2$

Exercise xxxix., page 100.

1. $2(x-1)(x^2-9x+10)$; $(x-2y)^2(x-3y)$.
2. $(4x+3y)(3x^2-xy+y^2)$; $(x-1)(4x-2)(2x+3)$.
3. $(x-5a)(3x^2+a^2)$; $(2x+3y)(x^2+3xy-y^2)$.
4. $(b+c)(b-4c)(2b^2-bc+c^2)$; $(5a+4b)(3a^2+7ab-3b^2)$.
5. $(2p+q)(2p+3q)(p^2+q^2)$.
6. $(10x-9y)(15x+16y)(x^2-5xy+8y^2)$.
7. $(2x-3y)(2x+3y)(3x+4y)(3x-5y)$.
$(5x-2z)(2x^3-3x^2y+3xy^2+12y^3)$.

CHAPTER IV.

Exercise xl., page 103.

1. Dividend $= (1-x) + x^2(1-x)$
   $\therefore$ quotient $= 1 + x^2$.

2. Dividend $= (x^4-1)^2 + (x^2+1)^2(x^2-1)^2$,
   divisor $= (x^2+1)^2$,
   $\therefore$ quotient $= (x^2-1)^2$.

3. Dividend $= (x^4 - a^2x^2 + a^4)(x^4 + a^2x^2 + a^4)(x^8 - a^4x^4 + a^8)$
   $\therefore$ quotient $= (x^4 + a^2x^2 + a^4)(x^8 - a^4x^4 + a^8)$.

4. Dividend $= (x-2y)(x+2y)(x^2+8y^2)$,
   $\therefore$ quotient $= (x+2y)(x^2+8y^2)$.

5. Dividend $= (1+2x-3x^2)(1-2x+3x^2)$,
   $\therefore$ quotient $= 1-2x+3x^2$.

6. Dividend $= (a^2-x^2)(a-x)(a+x)^2$,
   $\therefore$ quotient $= (a-x)(a+x)^2$.

7. Dividend $= (x-y+z)(x^2+y^2+z^2+xy+yz-zx)$,
   $\therefore$ quotient $= x^2+y^2+z^2+xy+yz-zx$.

8. Dividend $= (a+b)(3a+b)(2a^2-3ab+4b^2)$,
   $\therefore$ quotient $= (a+b)(3a+b)$.

9. Dividend $= (x-y)(2x+3y)(2x^2-xy+8y^2)$,
   $\therefore$ quotient $= (x-y)(2x+3y)$.

10. Dividend $= (a^2-b^2)^2 - c^4$,
    $\therefore$ quotient $= a^2-b^2+c^2$.

11. Dividend $= (7a^2-3ab+2b^2)(3a^2-ab+b^2)$,
    $\therefore$ quotient $= 7a^2-3ab+2b^2$.

12. Dividend $=(2a^2+7a+3)(a-7)$,

$\therefore$ quotient $=a-7$.

13. Dividend $=-(a-b)(b-c)(c-a)(a+b+c)$,

$\therefore$ quotient $=-(a-b)(b-c)(c-a)$,

or $(a-b)(a-c)(b-c)$.

14. Dividend $=(x-a)^3+b^3$,

$\therefore$ quotient $=(x-a)^2-b(x-a)+b^2$.

15. Dividend $=(x^2+z^2)^2-(y^2+1)^2$,

$\therefore$ quotient $=x^2+z^2+y^2+1$.

16. Dividend $=x(x^2-ax+b)(x-c)$,

$\therefore$ quotient $=x(x^2-ax+b)$

17. Dividend $=x^2+y^2$.

18. Dividend $=\dfrac{x^8-y^8}{x+y}=\dfrac{(x-y)(x+y)(x^2+y^2)(x^4+y^4)}{(x+y)}$,

$\therefore$ quotient $=(x-y)(x^2+y^2)$.

19. Dividend $=(a^2-b^2)^2-(c^2+1)^2$,

$\therefore$ quotient $=a^2-b^2+c^2+1$.

20. Dividend $=(a^3-b^3-c^3)(a-2b+3c)$, see Exercise XXXIII.

$\therefore$ quotient $=a^3-b^3-c^3$.

21. Dividend $=b(a^2-x^2)+x(a^2-x^2)$,

$=(x+b)(a-x)(a+x)$,

$\therefore$ quotient $=a+x$.

22. Dividend $=-(a-b)(a-c)(b-c)(a+b+c)$,

divisor $=(a-b)(a-c)$,

$\therefore$ quotient $=-(b-c)(a+b+c)$.

23. Dividend $=a^2b^2-c^2(a+b)^2$,

divisor $=(ab-ca-bc)(ab-ca+bc)$,

$\therefore$ quotient $=ab-ca-bc$.

24. Dividend $=(x+y-1)(x^2+y^2+1-xy+x+y)$,

$\therefore$ quotient $=x^2+y^2+1-xy+x+y$.

25. Dividend $=(x^3-2)(x^3+1)=(x^3-2)(x+1)(x^2-x+1)$,

$\therefore$ quotient $=(x^3-2)(x+1)$.

26. Dividend $=(a^2-5a-7)(a^2+5a+3)$,

$\therefore$ quotient $=a^2+5a+3$.

27. Dividend $=\{(2x-y)a^2-(x+y)ax+x^3\}$
$$\times\{(2x-y)a^2+(x+y)ax-x^3\},$$

$\therefore$ quotient $=(2x-y)a^2-(x+y)ax+x^3$.

28. Dividend $=\{(x-1)a^2-(x-1)a+3\}\{(x^2+x+1)a-(x+1)\}$,

$\therefore$ quotient $=(x^2+x+1)a-(x+1)$.

Exercise xli., page 107.

1. x^2-3. 2. $x+a$.

3. Factor as in Art. XXIII., and the C.M. $=x^2-x+1$.

4. Factor as in last Example, ax^2+bx+c.

5. No Common Measure: as may be found by adding and subtracting.

6. c^a+c^b. 7. $(a-b)(x+a)$. 8. $b(x+y)$.

9. $(a-b)(b-c)(c-a)$. 10. $a^{2m}+1$.

11. The C. F. must measure x^3+ax^2+bx+c
$$-x(x^2+a'x+b')=x^2(a-a')+x(b-b')+c.$$

12. $a-b$ is seen to be a factor of 1st, $\therefore$ also $b-c$, $c-a$ are factors; so writing a^2 for a, &c., in 2nd expn., its factors are a^2-b^2, b^2-c^2, c^2-a^2. It is also seen that 5 is a C. F. $\therefore$ $5(a-b)(b-c)(c-a)$ is the H. C. F. The *quadratic* factor may be discovered as on page 229, Hand-Book.

13. First quantity
$$=(y^2-1)(x-1)\{2(y-2)x^2+(2y-1)x+2y-1\};$$
2nd quantity $=(y-1)^2(x-1)\{3(y-2)x+3y+1\}$,

$\therefore$ H. C. F. is $(y-1)(x-1)$.

14. Let $x+a$ be the C. F.

Then $a^2-pa+q=0$ and $a^2-ma+n=0$, $\therefore$ $(m-p)a-n+q=0$,

$\therefore$ $a=\dfrac{n-q}{m-p}$, substitute this value of a, &c.

15. $x^3-3x^2+3x-1=(x-1)^3$, $x^3-x^2-x+1=(x+1)(x-1)^2$.
$x^4-2x^3-2x-1=(x^4-1)-2x(x^2-1)$
$=(x^2-1)(x^2-2x+1)=(x-1)^3(x+1)$
$x^4-2x^3+2x^2-2x+1=(x^4+2x^2+1)-2x(x^2+1)$
$=(x^2+1)^2-2x(x^2+1)=(x^2+1)(x^2-2x+1)$
$=(x^2+1)(x-1)^2$. $\therefore$ L. C. M. $=(x^2+1)(x+1)(x-1)^3$.

16. $x^3+6x^2+11x+6=(x+1)(x+2)(x+3)$
$x^3+7x^2+14x+8=(x+1)(x+2)(x+4)$
$x^3+8x^2+19x+12=(x+1)(x+3)(x+4)$
$x^3+9x^2+26x+24=(x+2)(x+3)(x+4)$.

All these factors may be easily found by substitution.

$\therefore$ L. C. M. is $(x+1)(x+2)(x+3)(x+4)$.

17. Denote first quantity by a, second by b; then $b-a\times xy=(y^2+y)x+5y-1\ldots\ldots r_1$. $a-r_1\times 2x=xy+4\ldots\ldots r_2$. $r_1-r_2(y+1)=y-5\ldots\ldots r_3$, which must $=0$, $\therefore y=5$.

18. The L. C. M. of a and $b=ab\div$ F.

(F = H. C. F.) $=\frac{1}{2}(a^2+b^2)$, $\therefore a^2+b^2=2ab$, or $a^2+b^2-2ab=(a-b)^2=0$. $\therefore a=b$. $\therefore$ the other quantity is same as given quantity.

19. Since $x+a$ is a measure, and also $x-a$,

$$-a^3+pa^2-qa+r=0, \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{and } a^3+pa^2+qa+r=0.$$

Adding, $2pa^2+2r=0$, $\therefore a^2=-\dfrac{r}{p}$.

Substituting in (1)

$$+\frac{ar}{p}-1-qa+r=0,\ ar=pqa,\ r=pq.$$

20. Since $x^2-2ax+a^2$ is a factor of both expressions, therefore if $2ax-a^2$ be substituted for x^2 both expressions will vanish.

Therefore, $2ax^2-a^2x+qx+r=0$,

$$4a^2x-2a^3-a^2x+qx+r=0,$$

also, $4a^2x-2a^3-a^2x+mx+n=0$.

$\therefore\ (m-q)x+n-r=0$ for all values of x,

$\therefore\ m-q=0$ and $n-r=0$, $\therefore\ m=q$ and $n=r$,

$\therefore\ q^2n^3=m^2r^3$. Note that the expressions are proved identical.

21. Let $x+b$ be the other factor of x^2+mx+n.

$$\text{Then } (x+a)(x+b)=x^2+mx+n$$

$$x^2+(a+b)x+ab=x^2+mx+n,$$

$$\therefore\ a+b=m;\ \text{or } b=(m-a)\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, L.C.M of x^3+px+q, and x^2+mx+n

$$=\frac{(x^3+px+q)(x+a)(x+b)}{x+a}$$

$$=(x^3+px+q)(x+b)=x^4+bx^3+px^2+(bp+q)x+bq\ldots(2).$$

Also, since $x+a$ is a factor of x^3+px+q,

$$\therefore\ -a^3-pa+q=0,$$

$$\therefore\ q=a^3+pa$$

$$bp+q=mp-pa+a^3+pa=a^3+mp$$

$$bq\quad=a(m-a)(a^2+p),$$

$\therefore$ (1) becomes $x^4+(m-a)x^3+px^2+(a^3+mp)x+a(m-a)(a^2+p)$

22. Since $x+a$ is a factor of x^2+qx+1

and also of x^3+px^2+qx+1

$$\therefore\ a^2-qa+1=0\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{and } -a^3+pa^2-qa+1=0\ \therefore\ a^3-(p-1)a^2=0$$

$$\therefore\ a=(p-1).$$

Substituting in (1),

$$(1-p)^2-q(1-p)+1=0$$

$$\text{or } (p-1)^2q-(p-1)+1=0.$$

23. From 21, if we let $(x+b)$ be the other factor of x^2+mx+n. The L.C.M. in this question will be $(x^3+px^2+q)(x+b)$

$$=x^4+(b+p)x^3+bpx^2+qx+bq.$$

Also as in 21. $b=m-a$.

But since $x+a$ is a factor of x^3+px^2+q

$$\therefore\ -a^3+pa^2+q=0$$
$$\therefore\ q=a^3-pa^2=a^2(a-p).$$

Substituting these values for b and q.

$$x^4+(b+p)x^3+bpx^2+qx+bq \text{ becomes}$$
$$x^4+(m-a+p)x^3+p(m-a)x^2+a^2(a-p)x$$
$$+a^2(m-a)(a-p).$$

24. Since $x-a$ is a factor of both x^2+px+1 and x^3+px^2+qx+1
$\therefore\ a^2+pa+1=0$............(1) and $a^3+pa^2+qa+1=0$.

Multiplying (1) by a, $a^3+pa^2+a=0$.

$$\therefore\ a(q-1)+1=0, \qquad \therefore\ a(q-1)=-1,$$
$$a=\frac{-1}{q-1}=\frac{1}{1-q}.$$

25. $(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3$
$=3(a^2-b^2)(b^2-c^2)(c^2-a^2)$
$=3(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)$.

But $a-b$, $b-c$, $c-a$ are factors of the other expression; also $(a+b)$ is *not* a factor. $\therefore$ also $(b+c)$ and $(c+a)$ are not.

$\therefore$ H. C. F. is $(a-b)(b-c)(c-a)$.

26. α, β, γ, each contain δ as a factor. $\frac{bc}{\delta^2}$, $\frac{ca}{\delta^2}$, and $\frac{ab}{\delta^2}$, have no common factor, $\therefore$ $\frac{bc}{\delta^2}\div\frac{\alpha\beta\gamma}{\delta^3}$ or $\frac{bc\delta}{\alpha\beta\gamma}$, $\frac{ca}{\delta^2}\div\frac{\alpha\beta\gamma}{\delta^3}$ or $\frac{ca\delta}{\alpha\beta\gamma}$, and $\frac{ab}{\delta^2}\div\frac{\alpha\beta\gamma}{\delta^3}$ or $\frac{ab\delta}{\alpha\beta\gamma}$, have no common factor, $\therefore$ L.C.M. $=$

$$\frac{\text{L.C.M. of numerators}}{\alpha\beta\gamma}=\frac{abc\delta}{\alpha\beta\gamma}.$$

27. $\left.\begin{array}{l} x^2+ax+b=(x+c)(x+a-c) \\ x^2+a'x+b'=(x+c)(x+a'-c) \end{array}\right\}$ $\therefore$ L.C.M. $=$ $(x+c)(x+a-c)(x+a'-c)$.

28. H. C. F. is $1-x$.

$\therefore$ L. C. M. $=(1+y+z+yz)(1-y-z+yz)(1-x)$
$=\{(1+yz)+(y+z)\}\{(1+yz)-(y+z)\}(1-x)$
$=(1-y^2-z^2+y^2z^2)(1-x)$.

Now substitute for x.

$$x = y^2+z^2-y^2z^2, \text{ and the L. C. M. becomes}$$

$$(1-y^2-z^2+y^2z^2)(1-y^2-z^2+y^2z^2) = (1-y^2-z^2+y^2z^2)^2.$$

29. $x^8+2x^6+3x^4-2x^2+1 = (x^4+x^2+1)^2-4x^2$

$$= (x^4+x^2+2x+1)(x^4+x^2-2x+1).$$

If x^4+x^3+2x+1 be a factor of $6x^8+x^7+17x^5-7x^3-2$, we see at once that $6x^4$ must be the other term; then to get x^7 the second term must be x^3: also there is no x^6, $\therefore$ the third term must be $-6x^2$; also $x^3+1=x^3$ and $-6x^2 \times 2x = -12x^3$. $\therefore$ to get $-7x^3$ the next term must be $4x$ and the last term -2.

$\therefore$ $x^4+x^3-6x^2+4x-2$ other factor of second expression.

$\therefore$ H. C. F. $= x^4+x^2+2x+1$.

Exercise xlii., page 111.

1. $$\frac{x^2-7x+6}{x^3-2x^2-8x-96} = \frac{(x-1)(x-6)}{(x^2+4x+16)(x-6)} = \frac{x-1}{x^2+4x+16};$$

$$\frac{3xy^2-13xy+14x}{7y^3-17y^2+6y} = \frac{x(3y-7)(y-2)}{y(7y-3)(y-2)} = \frac{x(3y-7)}{y(7y-3)}.$$

2. $$\frac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4} = \frac{(x^2-ax+a^2)(x^2+ax+a^2)}{(x+a)(x-a)(x^2+ax+a^2)}$$

$$= \frac{x^2-ax+a^2}{(x+a)(x-a)}; \quad \frac{x^2+x-12}{x^3-5x^2+7x-3} = \frac{(x+4)(x-3)}{(x-1)^2(x-3)} = \frac{x+4}{(x-1)^2}.$$

3. $$\frac{x^3-3x+2}{x^3+4x^2-5} = \frac{(x-1)(x-1)(x+2)}{(x-1)(x^2+5x+5)} = \frac{(x-1)(x+2)}{x^2+5x+5};$$

$$\frac{x^4+2x^2+9}{x^4-4x^3+4x^2-9} = \frac{(x^2+3)^2-4x^2}{x^2(x-2)^2-9} = \frac{(x^2+2x+3)(x^2-2x+3)}{(x^2-2x+3)(x^2-2x-3)}$$

$$= \frac{x^2+2x+3}{x^2-2x-3}.$$

4. $$\frac{2+bx}{2b+(b^2-4x)-2bx^2} = \frac{2+bx}{(b-2x)(2+bx)} = \frac{1}{b-2x};$$

$$\frac{x^3+2x^2+2x}{x^5+4x} = \frac{x(x^2+2x+2)}{x\{(x^2+2)^2-4x^2\}} = \frac{x^2+2x+2}{(x^2-2x+2)(x^2-2x+2)}$$

$$= \frac{1}{x^2-2x+2}.$$

5. $$\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4} = \frac{5a^3(a+x)^2}{x\{a^3+x^3+2ax(a+x)\}}$$

$$= \frac{5a^3(a+x)^2}{x(a+x)(a^2+ax+x^2)} = \frac{5a^3(a+x)}{x(a^2+ax+x^2)}$$

$$\frac{20x^4+x^2-1}{25x^4+5x^3-x-1} = \frac{(4x^2+1)(5x^2-1)}{(5x^2+x+1)(5x^2-1)} = \frac{4x^2+1}{5x^2+x+1}.$$

6. $$= \frac{x^4(x^3-x^2y+xy^2-y^3)+y^4(x^3-x^2y+xy^2-y^3)}{x^4(x^3+x^2y+xy^2+y^3)+y^4(x^3+x^2y+xy^2+y^3)}$$

$$= \frac{(x^4+y^4)(x^2+y^2)(x-y).}{(x^4+y^4)(x^2+y^2)(x+y)} = \frac{x-y}{x+y}.$$

Or thus, numerator $= \dfrac{x^8-y^8}{x+y}$, the denominator $= \dfrac{x^8-y^8}{x-y}$

$$\therefore \text{ given fraction } = \frac{x^8-y^8}{x+y} \cdot \frac{x-y}{x^8-y^8} = \frac{x-y}{x+y}.$$

7. $$\frac{3a^2x^4-2ax^2-1}{4a^3x^6-2a^2x^4-3ax^2+1} = \frac{(3ax^2+1)(ax^2-1)}{2ax^2(2a^2x^4-ax^2-1)-(ax^2-1)}$$

$$= \frac{(3ax^2+1)(ax^2-1)}{(4a^2x^4+2ax^2-1)(ax^2-1)} = \frac{3ax^2+1}{4a^2x^4+2ax^2-1};$$

$$\frac{x^2+\left(\dfrac{a}{b}+\dfrac{b}{a}\right)xy+y^2}{x^2+\left(\dfrac{a}{b}-\dfrac{b}{a}\right)xy-y^2} = \frac{\left(x+\dfrac{a}{b}y\right)\left(x+\dfrac{b}{a}y\right)}{\left(x+\dfrac{a}{b}y\right)\left(x-\dfrac{b}{a}y\right)} = \frac{ax+by}{ax-by}.$$

8. $$\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{abc(a-b)(b-c)(c-a)}.$$

The numerator vanishes when $a-b=0$. $\therefore$ $a-b$ is a factor; also $b-c$ and $c-a$ are factors, and the numerical coefficient is -1. $\therefore$ numerator $= -(a-b)(b-c)(c-a)$.

$$\frac{-(a-b)(b-c)(c-a)}{abc(a-b)(b-c)(c-a)} = -\frac{1}{abc}.$$

9. $$\frac{(a+b+c)^2}{a^3(b-c)+b^3(c-a)+c^3(a-b)} = \frac{(a+b+c)^2}{-(a-b)(b-a)(c-a)(a+b+c)}$$

$$= -\frac{a+b+c}{(a-b)(b-c)(c-a)}.$$

10. Let $-x=b-c$, $-y=c-a$ and consequently $x+y=a-b$. Substitute these values in the results of Ex. 4 referred to, then

$$\frac{2(x^2+xy+y^2)}{5xy(x+y)} = \frac{x(x+y)+y^2+x^2+(x+y)y}{5xy(x+y)}$$

$$\frac{(x+y)^2+x^2+y^2}{5xy(x+y)} = \frac{(a-b)^2+(b-c)^2+(c-a^2)}{5(a-b)(b-c)(c-a)}.$$

11. $(x+y)^5-x^5-y^5=5xy(x+y)(x^2+xy+y^2)$;

$$\begin{aligned}(x+y)^7-x^7-y^7 &= 7xy(x^5+3x^4y+5x^3y^2+5x^2y^3+3xy^4+y^5)\\ &= 7xy\{(x^5+y^5)+3xy(x^3+y^3)+5x^2y^2(x+y)\}\\ &= 7xy(x+y)(x^4+2x^3y+3x^2y^2+2xy^3+y^4)\\ &= 7xy(x+y)(x^2+xy+y^2)^2,\end{aligned}$$

$$\therefore \frac{(x+y)^5-x^5-y^6}{(x+y)^7-x^7-y^7} = \frac{5xy(x+y)(x^2+xy+y^2)}{7xy(x+y)(x^2+xy+y^2)^2} = \frac{5}{7(x^2+xy+y^2)}$$

12. This may be inferred from Ex. 11, in same manner as Ex. 10, from Ex. 4.

Exercise xliii., page 113.

1. $\dfrac{1-\frac{1}{2}\{1-\frac{1}{3}(1-x)\}}{1-\frac{1}{3}\{1-\frac{1}{2}(1-x)\}}$ multiplying both numerator and denominator by 6 $\therefore$ $\dfrac{6-3+(1-x)}{6-2+(1-x)} = \dfrac{4-x}{5-x}$;

$\dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$ Multiplying numerator and denominator by a^2-b^2; $\therefore$ $\dfrac{(a+b)^2+(a-b)^2}{(a+b)^2-(a-b)^2} = \dfrac{a^2+b^2}{2ab}$.

2. $$\frac{\dfrac{x}{x+y}+\dfrac{x}{x-y}}{\dfrac{2x}{x^2-y^2}} = \frac{(x^2-y^2)\left\{\dfrac{x}{x+y}+\dfrac{x}{x-y}\right\}}{(x^2-y^2)\left\{\dfrac{2x}{x^2-y^2}\right\}}$$

$$= \frac{x(x-y)+x(x+y)}{2x} = \frac{2x^2}{2x} = x;$$

$$\frac{\dfrac{1}{1-a}-\dfrac{1}{1+a}}{\dfrac{a}{1-a}+\dfrac{1}{1+a}}=\frac{(1-a^2)\left\{\dfrac{1}{1-a}-\dfrac{1}{1+a}\right\}}{(1-a^2)\left\{\dfrac{a}{1-a}-\dfrac{1}{1+a}\right\}}=\frac{2a}{a^2+1}$$

3. $$\frac{1}{1+\dfrac{a}{1+a+\dfrac{2a^3}{1+a}}};\quad \frac{a}{1+a+\dfrac{2a^2}{1+a}}=\frac{a(1+a)}{1+2a+3a^2}$$

$$\frac{1}{1+\dfrac{a(1+a)}{1+2a+3a^2}}=\frac{1+2a+3a^2}{1+3a+4a^2};\ \text{2nd result}=x.$$

4. $$\frac{\dfrac{a^2+b^2}{2a^2}-\dfrac{2b^2}{a^2+b^2}}{\dfrac{a^2+b^2}{2b^2}-\dfrac{2a^2}{a^2+a^2}}=\frac{\dfrac{(a^2-b^2)^2}{2a^2(a^2+b^2)}}{\dfrac{(a^2-b^2)^2}{2b^2(a^2+b^2)}}=\frac{\dfrac{1}{2a^2}}{\dfrac{1^2}{2b}}=\frac{b^2}{a^2}$$

$$\frac{\dfrac{1}{a}+\dfrac{1}{ab^3}}{b-1+\dfrac{1}{b}}=\frac{ab^3\left(\dfrac{1}{a}-\dfrac{1}{ab^3}\right)}{ab^3\left(b-1+\dfrac{1}{b}\right)}=\frac{b^3+1}{ab^2(b^2-b+1)}=\frac{b+1}{ab^2}.$$

5. $$\frac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}=\frac{(a+b)(c-d)+(a-b)(c+d)}{(a+b)(c+d)+(a-b)(c-d)}$$

$$=\frac{a(c-d+c+d)+b(c-d-c-d)}{a(c+d+c-d)+b(c+d-c+d)}$$

$$=\frac{2(ac-bd)}{2(ac+bd)}=\frac{ac-bd}{ac+bd}$$

$$\frac{a+b+\dfrac{b^2}{a}}{a+b+\dfrac{a^2}{b}}=\frac{ab\left(a+b+\dfrac{b^2}{a}\right)}{ab\left(a+b+\dfrac{a^2}{b}\right)}=\frac{b(a^2+ab+b^2)}{a(ab+b^2+a^2}=\frac{b}{a}.$$

6. Multiply numerator and denominator of second fraction by xyz, then it becomes

$$\frac{3xyz-(xy+yz+zx)}{xy+yz+zx} = \frac{3xyz}{xy+yz+zx} - 1$$

$$\frac{3xyz}{yz-zx-xy} - \frac{3xyz}{xy+yz+zx} + 1$$

$$\frac{3xyz(xy+yz+zx-yz+zx+xy)}{y^2z^2-x^2(y+z)^2} + 1 = \frac{6x^2yz(y+z)}{y^2z^2-x^2(y+z)^2} + 1.$$

7. Multiply both numerator and denominator by $a^2b^2c^2$, the fraction then becomes

$$\frac{2b^2c^2+2c^2a^2+2a^2b^2+a^4+b^4+c^4}{abc(a^2+b^2+c^2)} = \frac{(a^3+b^2+c^2)^2}{abc(a^2+b^2+c^2)}$$

$$= \frac{a^2+b^2+c^2}{abc}$$

8. $$\frac{a^3+a^2b+ab^2+b^3}{a^3-a^2b+ab^2-b^3} = \frac{\dfrac{a^4-b^4}{a-b}}{\dfrac{a^4-b^4}{a+b}} = \frac{a+b}{a-b}.$$

$$\frac{a^2+2ab+b^2}{a^2-b^2} = \frac{a+b}{a-b}, \qquad \frac{a+b}{a-b} \div \frac{a+b}{a-b} = 1.$$

9. $$\left(\frac{a+b}{a-b} + \frac{a^2+b^2}{a^2-b^2}\right) = \frac{(a+b)(a+b)+a^2+b^2}{a^2-b^2} = \frac{2(a^2+ab+b^2)}{a^2-b^2}$$

$$\frac{a-b}{a+b} - \frac{a^3-b^3}{a^3+b^3} = \frac{(a-b)(a^3+b^3)-(a+b)(a^3-b^3)}{(a+b)(a^3+b^3}$$

$$= \frac{a(a^3+b^3-a^3+b^3)-b(a^3+b^3+a^3-b^3)}{(a+b)(a^3+b^3)}$$

$$= \frac{-2ab(a^2-b^2)}{(a+b)(a^3+b^3)}$$

$$\frac{2(a^2+ab+b^2)}{a^2-b^2} \div \frac{-2ab(a^2-b^2}{(a+b)(a^3+b^3} = -\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}$$

10 $$\frac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}}=\frac{b+c+a}{b+c-a};\quad 1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c+a)(b+c-a)}{2bc}$$

$$\frac{b+c+a}{b+c-a}\times\frac{(b+c+a)(b+c-a)}{2bc}=\frac{(a+b+c)^2}{2bc}.$$

11. The first fraction $=\left\{\dfrac{\frac{1-x}{1+x}+1}{\frac{1+x}{1-x}+1}\right\}^2=\left\{\dfrac{\frac{2}{1+x}}{\frac{2}{1-x}}\right\}^2=\left(\dfrac{1-x}{1+x}\right)^2.$

The second fraction $=\left\{\dfrac{\frac{x-a}{x+a}-\frac{x+a}{x-a}}{\frac{x-a}{x+a}+\frac{x+a}{x-a}}\right\}^2=\left\{\dfrac{(x-a)^2-(x+a)^2}{(x-a)^2+(x+a)^2}\right\}^2$

$$=\frac{4a^2x^2}{(a^2+x^2)^2}.$$

12. $$\frac{\frac{x}{y}+1+\frac{y}{x}}{\frac{x}{y}-1+\frac{y}{x}}\div\frac{1-\frac{y^3}{x^3}}{1+\frac{y^3}{x^3}}.$$

Cancel numerator of first fraction into numerator of second, &c.

Then $1\div\dfrac{1-\frac{y}{x}}{1+\frac{y}{x}}=\dfrac{x+y}{x-y}.$

13. The fraction $=\dfrac{\left(1-\frac{a-b}{a+b}\right)^3}{\left(\frac{a+b}{a-b}-1\right)^3}=\dfrac{\left(\frac{2b}{a+b}\right)^3}{\left(\frac{2b}{a-b}\right)^3}=\left(\dfrac{a-b}{a+b}\right)^3.$

14. The dividend $= \dfrac{\dfrac{x^6-y^6}{x+y}}{\dfrac{x^6-y^6}{x-y}} = \dfrac{x-y}{x+y}$

$$\frac{x-y}{x+y} \div \left(\frac{x-y}{x+y}\right)^2 = 1 \div \frac{x-y}{x+y} = \frac{x+y}{x-y}$$

15. The dividend =

$$\left(\frac{1+x}{1+x+x^2} + \frac{1-x}{1-x+x^2}\right) = \frac{2}{(1+x+x^2(1-x+x^2)}$$

The divisor =

$$\left(\frac{1+x}{1+x+x^2} - \frac{1-x}{1-x+x^2}\right) = \frac{2x^3}{(1-x+x^2)(1-x+x^2)}$$

∴ the quotient =

$$\frac{2}{(1+x+x^2)(1-x+x^2)} \div \frac{2x^3}{(1+x+x^2)(1-x+x^2)} = \frac{1}{x^3}$$

16. $= \dfrac{a}{n(a-b)} + \dfrac{b}{n(b-a)} = \dfrac{1}{n}.$

17. $= \sqrt{\left\{1-\dfrac{4b}{(1+b)^2}\right\}} = \pm\dfrac{1-b}{1+b}.$

18. $=1 + \dfrac{2\sqrt{(a-bx)}}{\sqrt{(a+bx)}-\sqrt{(a-bx)}} = \dfrac{\sqrt{a}(1+c+1-c)}{\sqrt{a}(1+c-1+c)} = \dfrac{1}{c}.$

Exercise xliv., page 117.

1. $(x-a)\div 5.$ 2. $a+b.$

3. $\dfrac{4a^3}{a^4-x^4} \times \dfrac{-4a^2x}{a^4-x^4} = 16a^5x \div (a^4-x^4)^2.$

4. $= \dfrac{a}{a+b} + \dfrac{b}{a-b} - \dfrac{a}{a-b} + \dfrac{b}{a+b} = 0.$

5. $= -2 - \dfrac{7}{x-2} + 3 - \dfrac{8}{x+2} - 1 + \dfrac{16x-4}{x^2-4} = \dfrac{1}{x+2}.$

6. $= \dfrac{1}{2a^2(a^2-x^2)} + \dfrac{1}{2a^2(a^2+x^2)} = 1 \div (a^4-x^4).$

7. $\frac{1}{2}\left\{\frac{(3x+2y)^2-(3x-2y)^2}{9x^2-4y^2}\right\} = \frac{12xy}{9x^2-4y^2}$.

8. $\frac{6x}{4x^2-1} - \frac{3x-1}{x(2x-1)} + \frac{x}{x(4x^2-1)} + \frac{1}{x(16x^4-1)}$

$= \frac{4x^2+2}{x(16x^4-1)}$.

9. $\frac{-4}{x+2} + \frac{9}{2(x+3)} = \frac{x-6}{2(x+2)(x+3)}$, to this add $-\frac{x-1}{(x+2)(x+3)}$, and result $= -\frac{x+4}{2(x+2)(x+3)}$, to this add $\frac{1}{2(x+1)}$, and result $= 1 \div (x+1)(x+2)(x+3)$.

10. First and second fractions $= 4(x^2+y^2) \div (x^2-y^2)$, to this add third fraction and result $= 16x^2y^2 \div (x^4-y^4)$, to this add last fraction, result $= 4(x^4+4x^2y^2+y^4) \div (x^4-y^4)$.

11. $(a-b)^3 \div (x+a)^2(x+b)^2$.

12. Combine first and last fractions (of dividend), and take result with second, and that result with third. $\therefore\ 16a^7x \div (a^8-x^8)$. Similarly, divisor $= 8a^6x^2 \div a^8-x^8$, $\therefore\ 2a \div x$.

13. $= 1 + \frac{10}{11x+8} - 1 - \frac{7}{18} = \frac{236-77x}{18(11x-8)}$.

14. $1 \div (a-b)$. 15. $15a(3a-x) \div (9a+2x)(a+3x)$.

16. $= \frac{10x-7}{(2x-5)(x-1)} - \frac{1}{(2x-7)(x-4)}$. 17. 2. 18. $y^n(y^n-x^n)$.

19. First and third combined $= (a-b)^{2n} + (a-b)^n + 1$.

Second and fourth combined $= -(a-b)^n + 1$.

$\therefore$ result $= (a-b)^{2n} + 2$.

20. Combine first and second, result $= 1 \div (x^2+a^2)(x^2+b^2)$. Combine this with third, result $= 0$.

21. Combine first three, result $= \frac{2x^2}{x^6-1}$, this with last, result $= 4x^2 \div (x^{12}-1)$.

22. $-(a^2+b^2)(a^2-ab+b^2) \div (a^2-b^2)(a^2+ab+b^2)$.

Exercise xlv., page 121.

1. $=\dfrac{x(x-2y)^3-y(y-2x)^3}{(x+y)^3}=x-y$. (See Exercise IX., Ex. 10).

2. $=\dfrac{a(a+2b)^3-b(b+2a)^3}{(a-b)^3}=a+b$.

3. L. C. M. of denominators is $(a-b)(b-c)c-a)$.

$\therefore$ First num. is a^2-b^2, second is b^2-c^2, third is c^2-a^2. $\therefore$ result $=0$.

4. $=\dfrac{-b+c-c+a-a+b}{(a-b)(b-c)(c-a)}=0$.

5. L. C. M. of denominators is $(a+b)(b+c)(c+a)$; first three fractions give numerator $=(a-b)(b+c)(c+a)$ + anal + anal; of which $a-b$ is found to be a factor, $\therefore$ $b-c$, and $c-a$ are factors, and it becomes $-(a-b)(b-c)(c-a)$, $\therefore$ result $=0$.

6. L. C. M. of denominators is $(a+b)(b-c)(c+a)(x+a)(x+b)$ $(x+c)$, result $=\{(a+b)(c+a)x^2+2(ab+bc+ca)ax-2a^2bc\}\div$

$(a+b)(c+a)(x+a)(x+b)(x+c)$.

7. $=(x-y)(y-z)(z-x)\div(x-y)(y-z)(z-x)=1$.

8. $=\dfrac{-a^3(b-c)-b^3(c-a)-c^3(a-b)}{(a-b)(b-c)(c-a)}=a+b+c$.

9. $=\{-bc(b-c)-ca(c-a)-ab(a-b)\}\div(a-b)(b-c)(c-a)=1$.

10. x^3-y^3. 11. 0. 12. $=-(a-b)(b-c)(c-a)\div$ $(a+b)(b+c)(c+a)$.

13. Sum of numerators $=-a^2(b-c)(x-b)(x-c)$ − anal − anal. which vanishes for $a=b$, $\therefore$ $a-b$, $b-c$, $c-a$ are factors, and the other factor is of the form mx^2+n. See Ex. 2 in Hand-book. $n=0$, $m=1$, $\therefore$ result is $x^2\div(x-a)(x-b)(x-c)$.

14. Numerator $=(x-y)(y-z)(z-x)$, $\therefore$ result is 1.

15. Numerator $=\{-(a+b)+(b-c)+(c+a)\}^2-0$.

16. $\dfrac{b(x+a-b)+ax}{ab+(b-a)(x-b)}$.

Exercise xlvi., page 127.

1. $$\frac{a^2-ab+b^2}{ab-4b^2} = \frac{\frac{a^2}{b^2}-\frac{a}{b}+1}{\frac{a}{b}-4}.$$

$$= \frac{\frac{c^2}{d^2}-\frac{c}{d}+1}{\frac{c}{d}-4} = \frac{c^2-cd+d^2}{cd-4d^2}.$$

2. $$\frac{a^2}{b^2}=\frac{c^2}{d^2}=\frac{a^2-c^2}{b^2-d^2}. \quad \frac{a}{b}=\frac{a-c}{b-d}=\frac{a+c}{b+d}$$

$$\therefore \frac{a^2}{b^2}=\left(\frac{a-c}{b-d}\right)^2=\left(\frac{a+c}{b+d}\right)^2 \therefore \frac{a^2-c^2}{b^2-d^2}=\left(\frac{a-c}{b-d}\right)^2=\left(\frac{a+c}{b+d}\right)^2$$

3. $$\frac{a^2}{b^2}=\frac{c}{d}=\frac{a^2+c^2}{b^2+d^2} \quad \therefore \quad \frac{a}{b}=\frac{c}{d}=\sqrt{\frac{a^2+c^2}{b^2+d^2}}.$$

4. $$\frac{x}{y}=\frac{3}{2}, \quad \frac{2x^3-x^2y+y^3}{x^2y+xy^2+2y^3}=\frac{\frac{2x^3}{y^3}-\frac{x^2}{y^2}+1}{\frac{x^2}{y^2}+\frac{x}{y}+2}$$

$$= \frac{\frac{27}{4}-\frac{9}{4}+1}{\frac{9}{4}+\frac{3}{2}+2} = \frac{22}{23}.$$

5. $$\frac{ma-nc-pe}{mb-nd-pf} = \frac{mb\left\{\frac{a}{b}\right\}-nd\left\{\frac{c}{d}\right\}-pf\left\{\frac{e}{f}\right\}}{mb-nd-pf}$$

$$= \frac{mb\left\{\frac{a}{b}\right\}-nd\left\{\frac{a}{b}\right\}-pf\left\{\frac{a}{b}\right\}}{mb-nd-pf}=\frac{a}{b}.$$

6. $\left\{\dfrac{a-mc+ne}{b-md+nf}\right\}^3 = \left\{\dfrac{b\left(\dfrac{a}{b}\right)-md\left(\dfrac{c}{d}\right)+nf\left(\dfrac{e}{f}\right)}{b-md+nf}\right\}^3$

$= \left\{\dfrac{b\left(\dfrac{a}{b}\right)-md\left(\dfrac{a}{b}\right)+nf\left(\dfrac{a}{b}\right)}{b-md+nf}\right\}^3 = \left\{\dfrac{a}{b}\ \dfrac{(b-md+nf)}{(b-md+nf)}\right\}^3 = \dfrac{a^3}{b^3}.$

7. $\left(\dfrac{1+x}{1-x}\right)\left(\dfrac{1-x+x^2}{1+x+x^2}\right) = \dfrac{b}{a}\left(\dfrac{1+x+x^2}{1-x+x^2}\right)\left(\dfrac{1-x+x^2}{1+x+x^2}\right)$

or $\dfrac{1+x^3}{1-x^3} = \dfrac{b}{a} \quad \therefore \dfrac{2x^3}{2} = \dfrac{b-a}{b+a}.$

8. $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}} = \dfrac{a}{1}$

$\dfrac{2\sqrt{(a+x)}}{2\sqrt{(a-x)}} = \dfrac{a+1}{a-1}.$ [See (5), p. 122].

$\dfrac{a+x}{a-x} = \dfrac{(a+1)^2}{(a-1)^2}$

$\dfrac{2x}{2a} = \dfrac{(a+1)^2-(a-1)^2}{(a+1)^2+(a-1)^2} = \dfrac{4a}{2(a^2+1)}$

$\therefore x = \dfrac{2a^2}{a^2+1}.$

9. $\dfrac{mx+a+b}{nx+a+c} = \dfrac{mx-c-d}{nx-b-d} = \dfrac{(mx+a+b)-(mx-c-d)}{(nx+a+c)-(nx-b-d)}$

$= \dfrac{a+b+c+d}{a+b+c+d} = 1$

$\therefore mx+a+b = nx+a+c,\ (m-n)x = c-b.\ \therefore x = \dfrac{b-c}{n-m}.$

10. $\dfrac{a-b}{ay+bx} = \dfrac{b-c}{bz+cx} = \dfrac{c-a}{cy+az} = \dfrac{a+b+c}{ax+by+cz}$

$= \dfrac{(a-b)+(b-c)+(c-a)+(a+b+c)}{(ay+bx)+(bz+cx)+(cy+az)+(ax+by+cz)}$

$= \dfrac{a+b+c}{(a+b+c)(x+y+c)} = \dfrac{1}{x+y+z}.$

11. Let each of given ratios $= m$; then

$$(a+b)+\frac{b+c}{2}+\frac{c+a}{3} = m\{(a-b)+(b-c)+(c-a)\}=0.$$

Clearing of fractions, $8a+9b+5c=0$.

12. $$\frac{\sqrt{(a)}+\sqrt{(a-x)}}{\sqrt{(a)}-\sqrt{(a-x)}} = \frac{1}{a}$$

$$\frac{2\sqrt{(a-x)}}{2\sqrt{(a)}} = \frac{1-a}{1+a} \quad \therefore \quad \frac{a-x}{a} = \left(\frac{1-a}{1+a}\right)^2.$$

13. Each ratio $= \dfrac{\text{difference of numerators}}{\text{difference of denominators}}$

$$= \frac{x^2-y^2+z(x-y)}{x-y} = x+y+z.$$

14. From the given ratios the value of xy is found to be $\dfrac{x+y+4}{2}$;

$$\frac{x^2+2x+1}{x^2-2x+3} = \frac{y^2+2y+1}{y^2-2y+3} = \frac{x^2-y^2+2(x-y)}{x^2-y^2-2(x-y)} = \frac{x+y+2}{x+y-2}$$

$$= \frac{x+y+4-2}{x+y+4-6} = \frac{\dfrac{x+y+4}{2}-1}{\dfrac{x+y+4}{2}-3} = \frac{xy-1}{xy-3}.$$

15. $$\frac{25x^2-16}{10x+8} = \frac{(5x+4)(5x-4)}{2(5x+4)} = \frac{5x-4}{2}$$

$$\frac{3(x^2-4)}{2(x-2)} = \frac{3(x+2)(x-2)}{2(x-2)} = \frac{3(x+2)}{2}$$

$$\therefore \frac{5x-4}{2} = \frac{3(x+2)}{2} \quad \therefore 5x-4=3x+6.$$

$2x=10$; $x=5$. $\dfrac{x-\frac{4}{5}}{x+2} = \dfrac{4\frac{1}{5}}{7} = \frac{3}{5}$.

16. $y = \dfrac{4bc}{b+c} \quad \therefore \dfrac{y}{2b} = \dfrac{2c}{b+c}, \quad \dfrac{y+2b}{y-2b} = \dfrac{b+3c}{c-b}$.

[See (6), p. 122. Similarly $\dfrac{y+2c}{y-2c} = \dfrac{3b+c}{b-c}$

$$\therefore \frac{y+2b}{y-2b} + \frac{y+2c}{y-2c} = \frac{b+3c}{c-b} - \frac{3b+c}{c-b} = \frac{2(c-b)}{c-b} = 2.$$

17. Let each of given ratios $=m$.

$$\text{Then } \frac{a^2+b^2}{4} = m(a^2-b^2)$$

$$\frac{b^2+c^2}{5} = m(b^2-c^2)$$

$$\frac{c^2+a^2}{6} = m(c^2-a^2)$$

$$\frac{a^2+b^2}{4} + \frac{b^2+c^2}{5} + \frac{c^2+a^2}{6} = m(a^2-b^2+b^2-c^2+c^2-a^2) = 0.$$

Multiply through by 60 ; then

$$15(a^2+b^2)+12(b^2+c^2)+10(c^2+a^2) = 0 = 25a^2+27b^2+22c^2.$$

18. $$\frac{a^2}{x^2-yz} = \frac{b^2}{y^2-zx} = \frac{c^2}{z^2-xy} = \frac{a^2+b^2+c^2}{x^2+y^2+z^2-xy-yz-zx}$$

again. $$\frac{a^2}{x^2-yz} = \frac{a^2x}{x^3-xyz} = \frac{b^2y}{y^3-xyz} = \frac{c^2z}{z^3-xyz}$$

$$= \frac{a^2x+b^2y+c^2z}{x^3+y^3+z^3-3xyz} = \frac{a^2x+b^2y+c^2z}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\therefore \frac{a^2+b^2+c^2}{x^2+y^2+z^2-xy-yz-zx} = \frac{a^2x+b^2y+c^2z}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\therefore a^2+b^2+c^2 = \frac{a^2x+b^2y+c^2z}{x+y+z}.$$

Clearing of fractions $a^2x+b^2y+c^2z = (a^2+b^2+c^2)(x+y+z)$.

19. Let each ratio $=m$, then $x = m(a+b)-c)$

$$\therefore (a-b)x = m\{(a^2-b^2)-(a-b)c\}$$

$$(b-c)y = m\{b^2-c^2-(b-c)a\}$$

$$(c-a)z = m\{c^2-a^2-(c-a)b\}$$

$$\therefore (a-b)x+(b-c)y+(c-a)z = m\{a^2-b^2-(a-b)c+b^2-c^2$$
$$-(b-c)a+c^2-a^2-(c-a)b\} = 0.$$

20. $\left(\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2}\right)^2 = \left\{\dfrac{b^2\left(\dfrac{a^2}{b^2}\right)+d^2\left(\dfrac{c^2}{d^2}\right)+f^2\left(\dfrac{e^2}{f^3}\right)}{b^2+d^2+f^2}\right\}^2$

$= \left\{\dfrac{b^2\left(\dfrac{a^2}{b^2}\right)+d^2\left(\dfrac{a^2}{b^2}\right)+f^2\left(\dfrac{a^2}{b^2}\right)}{b^2+d^2+f^2}\right\}^2 = \left\{\dfrac{\left(\dfrac{a^2}{b^2}\right)(b^2+d^2+f^2)}{b^2+d^2+f^2}\right\}^2 = \dfrac{a^4}{b^4}.$

again, $\dfrac{a^4}{b^4}=\dfrac{c^4}{d^4}=\dfrac{e^4}{f^4}=\dfrac{a^4+c^4+e^3}{b^4+d^4+f^4} \therefore \left\{\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2}\right\}^2 = \dfrac{a^4+c^4+e^4}{b^4+d^4+f^4}.$

21. Let each of given ratios $=m$.

Then $bx+ay=m(a-b)$

and $bx+ay+cy+bz+az+cx$

$=m(a-b+b-c+c-a)=0,$

$\therefore\ bx+ay+cy+bz+az+cx+(ax+by+cz)=ax+by+cz,$

$\therefore\ x(a+b+c)+y(a+b+c)+z(a+b+c)$, or

$(a+b+c)(x+y+z)=ax+by+cz.$

22. $\dfrac{x^3-5x^2a-5xa^2-a^3}{x^3+x^2a+xa^2+a^3} = \dfrac{(x-a)^3-2xa(x-a)}{(x+a)^3+2xa(x+a)}$

$= \dfrac{2xa(x-a)}{2xa(x+a)} = \dfrac{\text{sum of numerators.}}{\text{sum of denominators.}}$

$= \dfrac{(x-a)^3}{(x+a)^3}\ \left(\dfrac{x-a}{x+a}\right)^3 = \dfrac{x-a}{x+a}\ \therefore\ \left(\dfrac{x-a}{x+a}\right)^2 = 1$

$\dfrac{x-a}{x+a} = \pm 1,\ \therefore\ x=0$, or $a=0.$

NOTE.—If the sign before $5xa$ (in numerator of given fraction) be $-$, the result is 5.

23. Inverting each fraction,

$\dfrac{6(a+b)}{a-b} = \dfrac{5(b+c}{b-c} = \dfrac{10(c+a)}{c-a}.$

Let each of these ratios $=m$; then $6(a+b)+5(b+c)+10(c+a)$ $=m(a-b+b-c+c-a)=0.\quad \therefore\ 16a+11b+15c=0.$

24. $\dfrac{x^2+2xyz+y^2z^2}{y^2+2xyz+z^2x^2} = \dfrac{1-y^2}{1-x^2}$

$\therefore \dfrac{x^2-y^2+y^2z^2-z^2x^2}{y^2+2xyz+z^2x^2} = \dfrac{(1-y^2)-(1-x^2)}{1-x^2}.$

Dividing both by x^2-y^2,

$$\frac{1-z^2}{y^2+2xyz+z^2x^2} = \frac{1}{1-x^2}.$$

Clearing of fractions and transposing,

$1-x^2-y^2-z^2-2xyz=0, \quad \therefore x^2+y^2+z^2+2xyz=1.$

25. Let each ratio $=m$; then $a+b+c=$

$m(x-y)+m(y-z)+m(z-x)=m(x-y+y-z+z-x)=0$

26. $\dfrac{a^2}{b^2} = \dfrac{ac}{bd} \quad \therefore \dfrac{a}{b} = \dfrac{\sqrt{(ac)}}{\sqrt{(bd)}}$

$\therefore \dfrac{a+b}{a-b} = \dfrac{\sqrt{(ac)}+\sqrt{(bd)}}{\sqrt{(ac)}-\sqrt{(bd)}}$ [See (6), p. 122].

27. $\dfrac{la+mc+ne}{lb+md+nf} = \dfrac{lb\left(\dfrac{a}{b}\right)+md\left(\dfrac{c}{d}\right)+nf\left(\dfrac{e}{f}\right)}{lb+md+nf}$

$= \dfrac{lb\left(\dfrac{a}{b}\right)+md\left(\dfrac{a}{b}\right)+nf\left(\dfrac{a}{b}\right)}{lb+md+nf} = \dfrac{\dfrac{a}{b}(lb+md+nf)}{(lb+md+nf)} = \dfrac{a}{b}.$

If $\dfrac{x}{2a+2b-c} = \dfrac{y}{2b+2c-a} = \dfrac{z}{2c+2a-b}$,

then $\dfrac{2a+2b-c}{x} = \dfrac{2b+2c-a}{y} = \dfrac{2c+2a-b}{z}$

$$= \frac{2(2a+2b-c)+2(2b+2c-a)-(2c+2a-b)}{2x+2y-z}$$

$$= \frac{2(2b+2c-a)+2(2c+2a-b)-(2a+2b-c)}{2y+2z-x}$$

$$= \frac{2(2c+2a-b)+2(2a+2b-c)-(2b+2c-a)}{2z+2x-y}$$

$$\therefore \quad \frac{9b}{2x+2y-z} = \frac{9c}{2y+2z-x} = \frac{9a}{2z+2x-y}$$

$$\therefore \quad \frac{b}{2x+2y-z} = \frac{c}{2y+2z-x} = \frac{a}{2z+2x-y}.$$

28. $\dfrac{a}{b} = \dfrac{c}{d} \quad \therefore \quad \dfrac{a}{c} = \dfrac{b}{d} = \dfrac{a-b}{c-d},$

$$\therefore \quad \left\{\frac{a}{c}\right\}^n = \left\{\frac{a-b}{c-d}\right\}^n; \text{ but } \sqrt{\left\{\frac{a^{2n}+b^{2n}}{c^{2n}+d^{2n}}\right\}}$$

$$= \sqrt{\left\{\frac{c^{2n}\left(\frac{a^{2n}}{c^{2n}}\right)+d^{2n}\left(\frac{b^{2n}}{d^{2n}}\right)}{c^{2n}+d^{2n}}\right\}} = \sqrt{\left\{\frac{c^{2n}\left(\frac{a^{2n}}{c^{2n}}\right)+d^{2n}\left(\frac{a^{2n}}{c^{2n}}\right)}{c^{2n}+d^{2n}}\right\}}$$

$$= \sqrt{\left\{\frac{\frac{a^{2n}}{c^{2n}}(c^{2n}+d^{2n})}{(c^{2n}+d^{2n})}\right\}} = \sqrt{\left(\frac{a^{2n}}{c^{2n}}\right)} = \frac{a^n}{c^n}$$

$$\therefore \quad \left(\frac{a-b}{c-d}\right)^n = \sqrt{\left(\frac{a^{2n}+b^{2n}}{c^{2n}+d^{2n}}\right)}.$$

29. Let each ratio $=r$; then

$$\frac{x}{a} = r(y+z) \text{ and } \frac{x}{a}(y-z) = r(y^2-z^2),$$

Similarly $\dfrac{y}{b}(z-x) = r(z^2-x^2)$ and $\dfrac{z}{c}(x-y)$

$$= r(x^2-y^2) \quad \therefore \quad \frac{x}{a}(y-z) + \frac{y}{b}(z-x) + \frac{z}{c}(x-y)$$

$$= r(x^2-z^2+z^2-x^2+x^2-y^2) = 0.$$

30. Let each ratio $=r$; then

$$\frac{a}{lx} = r(ny-mz) \quad \therefore \quad \frac{a}{lx}(l-x) = r(ny-mz)(l-x)$$

$$\frac{b}{my}(m-y) = r(lz-nx)(m-y) \text{ and}$$

$$\frac{c}{nz}(n-z) = r(mx-ly)(n-z)$$

$$\therefore \frac{a}{lx}(l-x)+\frac{b}{my}(m-y)+\frac{c}{nz}(n-z)$$

$$= r\{(ny-mz)(l-x)+(lz-nx)(m-y)+(mx-ly)(n-z)\}=0.$$

31. $z=\frac{\sqrt{(ay^2-a^2)}}{y} \quad \therefore y^2z^2=(ay^2-a^2)$

also, $x^2y^2=(ax^2-a^2)$,

$$\therefore \frac{y^2z^2}{x^2y^2}=\frac{a(y^2-a)}{a(x^2-a)} \quad \therefore \frac{z^2}{x^2}=\frac{y^2-a}{x^2-a};$$

Clearing of fractions

$$x^2y^2-ax^2=z^2x^2-az^2, \quad ax^2-a^2-ax^2=z^2x^2-az^2$$

$$\therefore z^2x^2=az^2-a^2 \quad \therefore x=\frac{\sqrt{(az^2-a^2)}}{z}$$

32. See solution of 18.

Or thus, $x^2+y^2+z^2-xy-yz-zx-a^2+b^2+c^2$

$$x^3-xyz+y^3-xyz+z^3-xyz=a^2x+b^2x+c^2z$$

that is $(x+y+z)(x^2+y^2+z^2-xy-xz-zx)=a^2x+b^2y+c^2z.$

Substitute value of $x^2+y^2+z^2-xy-xz-zy$,

then $(x+y+z)(a^2+b^2+c^2)=a^2x+b^2y+c^2x$, &c.

33. $\frac{m^2}{a^2}=\frac{m^2}{x^2}\cdot\frac{x^2}{a^2}=\frac{m^2}{x^2};\ \frac{n^2}{b^2}=\frac{n^2}{y^2}\cdot\frac{y^2}{b^2}=\frac{n^2}{y^2}$

$$\frac{r^2}{c^2}=\frac{r^2}{z^2}\cdot\frac{z^2}{c^2}=\frac{r^2}{z^2} \quad \therefore \frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{r^2}{c^2}$$

$$=\frac{m^2}{x^2}+\frac{n^2}{y^2}+\frac{r^2}{z^2}=\frac{3m^2}{x^2}=\frac{3(m^2+n^2+r^2)}{x^2+y^2+z^2}.$$

34. $\frac{a^{3n}}{b^{3n}}=\frac{c^{3n}}{d^{3n}}=\frac{a^{3n}-c^{3n}}{b^{3n}-d^{3n}};$

but $\frac{a^{3n}}{b^{3n}}=\frac{a^nc^ne^n}{b^nd^ne^u}$ and $\frac{a^n}{b^n}=\frac{c^n}{d^u}=\frac{e^n}{f^n}=\frac{a^n-c^n+e^n}{b^n-d^n+f^n}$,

$$\therefore \frac{a^{3n}}{b^{3n}}=\left(\frac{a^n-c^n+e^n}{b^n-d^u+f^n}\right)^3; \quad \frac{a^{3n}}{b^{3n}}=\frac{a^nc^ne^n}{b^nd^nf^n}$$

$$= \frac{(a^n - c^n + e^n)^3}{(b^n - d^n + f^n)^3} = \frac{a^n c^n e^n - (a^n - c^n + e^n)^3}{b^n d^n f^n - (b^n - d^n + f^n)^2}.$$

Hence the equality required.

35. $\frac{a_1^2}{b_2^2} = \frac{a_1 a_2}{b_1 b_2} = \frac{a_2 a_3}{b_2 b_3} = \frac{a_3 a_4}{b_3 b_4} = \frac{a_1 a_2 - a_2 a_3 + \ldots (-1)^{n-1} a_{n-1} a_n}{b_1 b_2 - b_2 b_3 + \ldots (-1)^{n-1} b_{n-1} b_n}$

also since $\frac{a_1^2}{b_1^2} = \frac{a_2 a_3}{b_2 b_3}$ $\therefore$ $\frac{a_1}{b_1} = \frac{\sqrt{(a_2 a_3)}}{\sqrt{(b_2 b_3)}}$ and $\frac{a_1^2}{b_1^2} = \frac{a_1 \sqrt{(a_2 a_3)}}{b_1 \sqrt{(b_2 b_3)}}$.

Similarly $\frac{a_1^2}{b_1^2} = \frac{a_2 \sqrt{(a_3 a_4)}}{b_2 \sqrt{(b_3 b_4)}} = \frac{a_3 \sqrt{(a_4 a_5)}}{b_3 \sqrt{(b_4 b_5)}} =$ &c.

$$\therefore \frac{a_1^2}{b_1^2} = \frac{a_1 \sqrt{(a_1 a_3)} + a_2 \sqrt{(a_3 a_4)} + \&c.}{b_1 \sqrt{(b_2 b_3)} + b_2 \sqrt{(b_3 b_4)} + \&c.}$$

Hence the equality required.

36. From first equation

$$A = -\frac{B(1-ca) + C(1-ab)}{1-bc}.$$

From second equation

$$A = -\frac{B(c+a) + C(a+b)}{b+c}$$

$$\therefore \frac{B(1-ca) + C(1-ab)}{1-bc} = \frac{B(c+a) + C(a+b)}{b+c}.$$

Hence $\frac{B(a-b)}{1+b^2} = \frac{C(c-a)}{1+c^2}$ (1)

Similarly $\frac{B(b-c)}{1+b^2} = \frac{A(c-a)}{1+a^2}$ (2)

$$\therefore \frac{B}{1+b^2}\left\{a-b+b-c\right\} = (c-a)\left\{\frac{A}{1+a^2} + \frac{C}{1+c^2}\right\}.$$

Transposing and dividing by $a-c$

$$\frac{A}{1+a^2} + \frac{B}{1+b^2} + \frac{C}{1+c^2} = 0.$$

Secondly, dividing both sides of (1) and (2) by a and c respectively, we get

$$\frac{B(a-b)}{a(1+b^2)} = \frac{C(c-a)}{a(1+c^2)} \text{ and } \frac{B(b-c)}{c(1+b^2)} = \frac{A(c-a)}{c(1+a^2)}$$

$$\therefore \frac{B}{1+b^2}\left\{\frac{a-b}{a} + \frac{b-c}{c}\right\} = (c-a)\left\{\frac{C}{a(1+c^2)} + \frac{A}{c(1+a^2)}\right\}$$

$$\frac{Bb}{1+b^2}\left\{\frac{a-c}{ac}\right\} = \frac{c-a}{ac}\left\{\frac{C}{1+c^2} + \frac{A}{1+a^2}\right\}$$

Transposing and dividing by $\dfrac{a-c}{ac}$ we get

$$\frac{Aa}{1+a^2} + \frac{Bb}{1+b^2} + \frac{Cc}{1+c^2} = 0.$$

Dividing both numerator and denominator by a, b and c, we get

$$\frac{A}{a+\dfrac{1}{a}} + \frac{B}{b+\dfrac{1}{b}} + \frac{C}{c+\dfrac{1}{c}} = 0.$$

37. $\dfrac{x^2}{a^2} \div \dfrac{xh}{a^2} = \dfrac{y^2}{b^2} \div \dfrac{yk}{b^2} = \dfrac{z^2}{c^2} \div \dfrac{zl}{c^2}$,

$$\therefore \frac{x}{h} = \frac{y}{k} = \frac{z}{l} = 1 \div \frac{xh}{a^2} = \frac{a^2}{xh},$$

$$\therefore \left(\frac{x}{h} + \frac{y}{k} + \frac{z}{l}\right)^2 = \left(\frac{3a^2}{xh}\right)^2 = \frac{9a}{x^2h^2};$$

again, $\dfrac{x^2}{a^2} \div \dfrac{x^2h^2}{a^4} = \dfrac{y^2}{b^2} \div \dfrac{y^2k^2}{b^4} = \dfrac{z^2}{c^2} \div \dfrac{z^2l^2}{c^4}$,

$$\therefore \frac{a^2}{h^2} = \frac{b^2}{k^2} = \frac{c^2}{l^2} = 1 \div \frac{x^2h^2}{a^4} = \frac{a^4}{x^2h^2} = \left(\frac{a^2}{xh}\right)^2,$$

$$\therefore \frac{a^2}{h^2} + \frac{b^2}{k^2} + \frac{c^2}{l^2} = \frac{3a^4}{x^2h^2}$$

and $\therefore \left(\dfrac{x}{h} + \dfrac{y}{k} + \dfrac{z}{l}\right)^2 = 3\left(\dfrac{a^2}{h^2} + \dfrac{b^2}{k^2} + \dfrac{c^2}{l^2}\right)$

Exercise xlvii., page 134.

1 $x^2+ab-(a+b)x-c^2 = x^2+2px+p^2$,

$\therefore\ 2p=(a+b), \qquad p^2=ab-c^2, \qquad \therefore\ 4p^2=(a+b)^2$,

$\therefore\ (a+b)^2=4(ab-c^2)$, or $(a-b)^2+4c^2=0$.

2. $4(2x^2n)=64x^2$; $\therefore\ n=8$.

3. Extracting square root (See Ex. 1 in Hand-Book), we find remainder to be $-3x+30$, which must $=0$; $\therefore\ x=10$.

4. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$

$$=(a-b)^4-2(a^2+b^2)(a-b)^2+(a^2+b^2)^2+(a^2-b^2)^2$$
$$=\{(a-b)^2-(a^2+b^2)\}^2+(a^2-b^2)^2$$
$$=4a^2b^2+(a^2-b^2)^2=(a^2+b^2)^2, \quad \therefore \text{ sq. root}=a^2+b^2.$$

5.

$$\begin{array}{lll}
 & & \underline{2x^2-x+1} \\
 & & 4x^4-4x^3+5x^2-mx+n \\
 & & \underline{4x^4\qquad\qquad\qquad\qquad} \\
4x^2-x\ ) & & \quad -4x^3+5x^2 \\
 & & \quad \underline{-4x^3+\ \ x^2} \\
4x^2-2x\ ) & & \qquad\qquad 4x^2-mx+n \\
 & & \qquad\qquad \underline{4x^2-2x+1} \\
 & & \qquad\qquad\quad (2-m)x+(n-1)
\end{array}$$

$\therefore\ (2-m)x+(n-1)=0$ for all values of x,

$\therefore\ 2-m=0$ or $m=2$ and $n-1=0$ or $n=1$.

6. Given expression $=$

$$x^4-2x^2+4-4x^2=x^4-6x^2+4=(x^2-2)^2-2x^2,$$

which is a perfect square if $2x^2$ be added; also x^4-6x^2+4 is a square if 5 be added.

7. $(x^2+mx+6)(x^2+nx+8)=x^4+x^3-16x^2-4x+48$

$$=x^4+(m+n)x^3+(mn+14)x^2+(8m+6n)x+48,$$

$\therefore\ m+n=1$, and $8m+7n=-4$, $\therefore\ m=-5$, $n=6$.

8. Extracting the square root as in example 5 above, the remainder is found to be

$$1+\left(\frac{80-c^2}{64}\right)\frac{cx}{2}+1-\left(\frac{80-c^2}{64}\right)^2$$

in which the coefficient of $x=0$, *and* the the last term $=0$; these conditions give $c=\pm 12$. The roots of the resulting expressions are $2x^2-3x+1$, and $2x^2+3x-1$.

9. Expression =
$4(a^2-b^2)^2c^2d^2+4a^2b^2(c^2-d^2)^2+(a^2-b^2)(c^2-d^2)^2+16a^2b^2c^2d^2$
$=(a^2-b^2)^2\{4c^2d^2+(c^2-d^2)^2\}+4a^2b^2\{(c^2-d^2)^2+4c^2d^2\}$
$=(a^2+b^2)^2(c^2+d^2)^2$, $\therefore$ square root is $(a^2+b^2)(c^2+d^2)$.

10. Applying condition of perfect square,
$(a+b)^4=4(a^2-b^2)(a+b)(a-b)$ $\therefore$ $(a+b)^2=4(a-b)^2$,
$\therefore$ $a+b=\pm 2(a-b)$, $\because$ &c.

11.

$$
\begin{array}{lll}
 & & ax+2b+2c \\
 & & \overline{a^2x^2+4abx+4acx+5bc+b^2c^2} \\
 & & a^2x^2 \\
2ax\ ) & & \overline{\quad 4abx+4acx+5bc+b^2c^2} \\
 & & \quad 4abx+4b^2 \\
2ax+4b+2c\ ) & & \overline{\qquad 4acx+5bc+b^2c^2-4b^2} \\
 & & \qquad \underline{4acx+8bc+4c^2} \\
 & & \qquad\quad -3bc-4c^2+b^2c^2-4b^2.
\end{array}
$$

12. Expression $=(x^2+mx+p)(x^2+nx+q)=$
$x^4+(m+n)x^3+(p+q+mn)x^2+(np+mq)x+pq$; equating coefficients, $\therefore$ $m+n=-4$..................(1)

$p+q+mn=-1$...........(2)

$np+mq\ =\ 16$...........(3). If p and q are rational integers, their values must be, since $pq=-12$, 1 and 12, or 2 and 6, or 3 and 4, one positive and the other negative. Now, if 3 and -4 be substituted for p and q respectively in (2) and (3), the resulting values of m and n, obtained from (3) and (1) are -4, and 0 respectively, and these values satisfy (2), $\therefore$ factors are $(x^2-4x+3)(x^2-4)$. By similar reasoning other values of p, q, m, n, may be found, x giving $(x^2-3x+2)(x^2-x-6)$ and also $(x^2+x-2)(x^2-5x+6)$.

13. Let $x^2+pax+a^2$ be the other factor.

Then $(x^2+pax+a^2)(x^2+max+a^2)=$

$$x^4+(m+p)ax^3+(mp+2)a^2x^2+(m+p)a^3x+a^4$$

$\therefore$ $m+p=-1$, and $mp+2=1$ or $mp=-1$.

$$\begin{array}{r} m^2+2mp+p^2=1 \\ 4mp \qquad =-4 \\ \hline m^2-2mp+p^2=5 \end{array}$$

$$m-p=\sqrt{5}, \quad \therefore m = \frac{\pm\sqrt{(5-1)}}{2}.$$

14. Let it be the square of $\left(x^2+\frac{a}{2}x+\frac{c}{a}\right)$.

Then $\left(x^2+\frac{a}{2}x+\frac{c}{a}\right)^2 = x^4+ax^3+bx^2+cx+d,$

i.e., $x^4+ax^3+\left(\frac{2c}{a}+\frac{a^2}{4}\right)x^2+cx+\frac{c^2}{a^2}$

$=x^4+ax^3+bx^2+cx+d$

$\therefore \frac{2c}{a}+\frac{a^2}{4}=b$, and $\frac{c^2}{a^2}=d,$

$\therefore 8c=a(4b-a^2), \qquad \therefore 64d=(4b-a^2)^2.$

15. $\{(x\sqrt{(a)}+y\sqrt{(b)}+z\sqrt{(c)}\}^2=$

$ax^2+by^2+cz^2+2xy\sqrt{(ab)}+2yz\sqrt{(bc)}+2zx\sqrt{(ac)}$

$\therefore 2\sqrt{(ab)}=d,\ 2\sqrt{(bc)}=e,\ 2\sqrt{(ac)}=f,$

$4ab=d^2, \qquad 4bc=e^2, \qquad 4ac=f^2,$

$$\frac{a}{c}=\frac{d^2}{e^2},\ \frac{a}{b}=\frac{f^2}{e^2},\ \frac{b}{c}=\frac{d^2}{f^2}.$$

16. Let $x+p$ be the other factor,

Then $(x+p)(x^2+2dx+d^2)=x^3+(2d+p)x^2+(d^2+2pd)x+pd^2$

$\therefore 2d+p=a,\ d^2+2pd=b. \quad pd^2=c$

$$pa=d(a-2d) \quad =\tfrac{1}{2}(b-d^2)=\frac{c}{d}.$$

17. Let $(x\sqrt[3]{a}-\sqrt[3]{d})^3=ax^3-bx^2+cx-d.$

$ax^3-3x^2\sqrt[3]{(a^2d)}+3x\sqrt[3]{(ad^2)}-d=ax^3-bx^2+cx-d,$

$\therefore 3\sqrt[3]{(a^2d)}=b, \quad 3\sqrt[3]{(2d^2)}=c$

$\therefore 27a^2d \ =b^3,\ 27ad^2=c^3$

$$\therefore \frac{a}{d}=\frac{b^3}{c^3},\ 9ad=bc.$$

18. The remainder on dividing is $(3a^3+6ab+3c)x+$
$d-3a^2b-2a^4$, $\therefore$ $3a^3+6ab+3c=0\ldots\ldots(1)$
and $\qquad 2a^4+3a^2b \quad =d\ldots\ldots(2)$.

From (1) $(a^2+b)^2=b^2-ac$; from (2) $d=2a^2(a^2+2b)-a^2b=$
$-2ac-a^2b$, $\therefore$ $c^2-bd=c^2+2abc+a^2b^2=(ab+c)^2=$
$(ab-a^3-2ab)^2=a^2(a^2+b)^2$, $\therefore$ $4(b^2-ac)(c^2-bd)=$
$4(ac-b^2)(bd-c^2)=4a^2(a^2+b)^2(a^2+b)^2=4a^2(a^2+b)^4$.
$ad=3a^3b+2a^5$ from (1); $-bc=a^3b+2ab^2$ from (2),
$\therefore$ $(ad-bc)^2=(2a^5+4a^3b+2ab^2)^2=4a^2(a^4+2a^2b+b^2)^2$
$=4a^4(a^2+b)^4$, $\therefore$ $(ad-bc)^2=4(b^2-ac)(c^2-bd)$.

19. Let $x^3+px^2+q=(x+m)(x^2-2ax+a^2)$
$=x^3+(m-2a)x^2+(a^2-2am)x+a^2m$,

$\therefore$ $m-2a \quad =p$, and $m=\dfrac{a}{2}$.

$a^2-2am \quad =0$, $\therefore$ $a=-\frac{2}{3}p$;

Also $a^3-2a^2m \quad =0$, or $a^3-2q=0$, $\therefore$ $a^3=2q$.

$\therefore$ $2q=-\frac{8}{27}p^3$, or $4p^3+27q=0$.

20. Let $x^2+max+a^2$ be the other factor, then
$x^4+ax^3+a^2x^2+a^3x+a^4=(x^2+nax+a^2(x^2+max+a^2)$
$=x^4+a(m+n)x^3+a^2(2+mn)x^2+a^3(m+n)x+a^4$.
$\therefore$ $m+n=1$, $2+mn=1$; from these equations $m-n=\pm\sqrt{5}$,
$\therefore$ $2n=1\pm\sqrt{5}$, $\therefore$ $n^2-n-1=0$.

21. Let $x^4+ax^3+bx^2+cx+d=(x+m)^2(x+n)^2=$
$x^4+2(m+n)x^3+(m^2+n^2+4mn)x^2+2mn(m+n)x+m^2n^2$.
$\therefore$ $2(m+n)=a$, $m^2+n^2+4mn=b$, $2mn(m+n)=c$,
$m^2n^2=d$; $\therefore$ $(4b-a^2)^2=\{4(m+n)^2+8mn-4(m+n)^2\}$
$=64m^2n^2=64d$. Also $(4b-a^2)a=16mn(m+n)=8c$.

22. Let $x^4+px^3+qx^2+rx+s=\left(x^2+\dfrac{p}{2}x+\sqrt{s}\right)^2=$

$$x^4+px^3+\left(\frac{p^2}{4}+2\sqrt{s}\right)x^2+p\sqrt{s}.x+s,$$

$$\therefore\ q = \frac{p^2}{4} + 2\sqrt{s},\ r = p\sqrt{s},\ \therefore\ r^2 = p^2 s.$$

23. Dividing by $ax^2+2bx+c$, the remainder is found to be

$$\left(2c - \frac{2b^2}{a}\right)x + d - \left(\frac{bc}{a}\right)$$

$$\therefore\ 2c - \frac{2b^2}{a} = 0,\ \text{or}\ ac = b^2 \ldots\ldots\ldots(1)$$

$$\text{and}\ d - \frac{bc}{a} = 0,\ \text{or}\ ad = bc \ldots\ldots\ldots(2).$$

From (1) $b = \sqrt{(ac)}$

$$\therefore\ ax^2+2bx+c = ax^2 + 2\sqrt{(ac)} + c = (x\sqrt{a} + \sqrt{c})^2.$$

$$(1)\times(2)\quad \therefore\ b^3c = a^2cd \quad \therefore\ b^3 = a^2d\ \text{and}\ b = a^{\frac{2}{3}}d^{\frac{1}{3}}$$

$$\text{Also}\quad \frac{a^2d^2}{ac} = \frac{b^2c^2}{b^2} \quad \therefore\quad ad^2 = c^3 \quad \therefore\ c = a^{\frac{1}{3}}d^{\frac{2}{3}}.$$

$$\therefore\ ax^3+3bx^2+3cx+d = ax^3+3a^{\frac{2}{3}}d^{\frac{1}{3}}x^2+3a^{\frac{1}{3}}d^{\frac{2}{3}}x+d = (a^{\frac{1}{3}}x+d^{\frac{1}{3}})^3.$$

24. $4m^2x^2(pq+q^2) = p^2x^2$, $\therefore\ p^2 - 4m^2qp - 4m^2q^2 = 0$,

$$\therefore\ p = 2m^2q \pm 2mq\sqrt{(m^2+1)}.$$

25. Let $x + \dfrac{r}{4}$ be the other factor, then $x^3+px^2+qx+r =$

$$(x+2)^2\left(x+\frac{r}{4}\right) = x^3 + \left(4+\frac{r}{4}\right)x^2 + (r+4)x + r.$$

$$\therefore\ p = 4 + \frac{r}{4};\ q = r+4$$

$$\therefore\ 4p = 16+r = q+12,\ \text{or}\ 4(p-3) = q.$$

26. If divisible by $3x^2+2px+q$, it is divisible by $x^2 - \frac{2}{3}px + \frac{1}{3}q$.

Let $x+m$ be the other factor; then $(x+m)(x^2+\frac{2}{3}px+\frac{1}{3}q) =$

$$x^3+px^2+qx+r = x^3+(m+\tfrac{2}{3}p)x^2+(\tfrac{1}{3}q+\tfrac{2}{3}mp)x+\tfrac{1}{3}mq;$$

$$\therefore\ m+\tfrac{2}{3}p = p,\ \text{or}\ m = \frac{p}{3}\quad x+m = x+\frac{p}{3}.$$

Exercise xlviii., page 137.

1. If $aa'=bb'=cc'$ then will $(a-b')a'=(b-a')b'$

$(b-c)b'=(c'-b')c$ and $(c'-a')c=(a-c)a'$

$\therefore\ (a-b')(b-c)(c'-a')=(b-a')(c'-b')(a-c)$.

Nos. 2 to 7 may be proved in like manner, or thus:

2. Interchange b and b' in 1 and it becomes—

If $aa'=b'b=cc'$ then will

$$(a-b)(b'-c)(c'-a')=(b'-a')(c'-b)(a-c)$$

which is 2 with the members transposed and the factors in different order.

3. Interchange a and a' in 1 and it becomes—

If $a'a=bb'=cc'$ then will

$$(a'-b')(b-c)(c'-a)=(b-a))(c'-b')(a'-c)$$

a mere variation of 3.

4. Interchange c and c' in 1 and it becomes—

If $aa'=bb'=c'c$ then will

$$(a-b')(b-c')(c-a')=(b-a')(c-b')(a-c')$$

which differs from 4 only in the order of the factors of the right-hand member.

5. Divide 4 by 2, member by member.

$$\frac{(a-b')(b-c')(c-a')}{(b-c')(c-a)(a'-b')}=\frac{(a-c')(b-a')(c-b')}{(c-b')(b-a)(a'-c')}$$

Rejecting factors common to both numerator and denominator,

$$\frac{(a-b')(c-a')}{(c-a)(a'-b')}=\frac{(a-c')(b-a')}{(b-a)(a'-c')}$$

from which 5 may be immediately obtained by a transferrence of factors.

6. Divide 4 by 3, member by member.

7. Divide 4 by 1, member by member.

6 and 7 may also be obtained from 5 by operating with the substitutions $(abc|a'b'c')$ and $(abc|a'b'c')^2$ respectively.

8. From 1 by actual multiplication

$$(ac'+a'b'-aa'-b'c')(b-c)=(ab+a'c-aa'-bc)(c'-b')$$

$$\therefore\ a(bc'-cc')+a'(bb'-b'c)-baa'+b'cc'+caa'-c'bb'=$$

$$a(bc'-bb')+a'(cc'-b'c)-bcc'+b'aa'+cbb'-c'aa'$$

$$\therefore\ (a+a')(bb'-cc')+(b+b')(cc'-aa')+(c+c')(aa'-bb')=0.$$

Hence the equation in 8 is merely another form of that in 1.

The equations in 2, 3, and 4 may be formed from that in 1 by interchanging b and b', a and a', and c and c' respectively, hence variations of them may be formed from the equation in 8 (proved to be a variation of that in 1) by the same system of interchanges. But these interchanges have no effect on the equation in 8, *i.e.*, they leave it unchanged, hence the equations in 2, 3 and 4 are merely different forms of that in 8; and consequently the equations in 1, 2, 3, 4 and 8 are all merely different ıorms of one and the same equation. Also 5, 6, and 7 have been shown to be formed from 1, 2, 3 and 4, and consequently they may all be formed from 8.

(The example and exercise of Art. XXXVIII. are important geometrical theorems—See Chasles' *Traite' de Geometrie superieure*).

CHAPTER V.

Exercise xlix., page 138.

1. $5,\ 3\frac{1}{4},\ a,\ -3.$ 2. $-4\frac{1}{2},\ -a,\ 2,\ 10.$

3. $a+b,\ c-a,\ b-c,\ 3.$ 4. $-2,\ 6,\ -5,\ 12.$

5. $-14,\ a-3b,\ 2a-3b,\ 5b-3a.$ 6. $7,\ 4,\ a,\ b.$

7. $\frac{c}{3},\ \frac{5}{a},\ 0,\ 1.$ 8. $-1,\ \frac{(a+b)^2-a}{b},\ a+b.$

9. $b-a,\ a+b.$ 10. $\frac{1}{a+b},\ \frac{1}{a-b},\ \frac{1}{a^2+b^2}.$ 11. $2b,\ a.$

12. $a+b,\ \frac{c}{a+b},\ \frac{b}{a-c}.$ 13. $\frac{b-c}{a-b},\ b+c.$

14. $a+b,\ a^2+ab+b^2.$ 15. $a^2-ab+b^2,\ 1.$

16. $-1,\ \frac{a+b}{a-b}.$ 17. $\frac{e+b}{e-b},\ \frac{2}{15},\ \frac{3}{14}.$

18. $-\frac{1}{12},\ \frac{b}{ac},\ \frac{a}{b}.$ 19. $\frac{a^2+b^2}{a^2b^2},\ \frac{a(b^3+c^3)}{bc}.$

20. $10,\ 12,\ 4,\ \frac{1}{5}.$ 21. $1000,\ \frac{3}{5},\ \frac{3}{2}.$

22. $9\frac{9}{10},\ ab,\ \frac{bc}{a}.$ 23. $\frac{b^2}{ac},\ c(a+b),\ \frac{b}{a}\ (a+b).$

24. $\frac{a}{b},\ \frac{a-b}{a+b},\ -\frac{(a+b)^2}{(a-b)^2}.$ 25. $-1,\ -1.$

26. $\frac{a^2-c^2)}{(a+b)^2},\ 2,\ \frac{10}{3}.$ 27. $ab,\ \frac{b}{a},\ \frac{ac}{b},\ 12.$

28. $12,\ \frac{-ac}{b}.$ 29. $9,\ 2.$

30. $12,\ 1.$ 31. $3,\ 1.$ 32. $\frac{2a-1}{2a+2},\ 0.$

33. $\frac{1}{m}.$ 34. $1.$ 35. $\frac{ab+bc+ca}{a^2+bc+c^2}.$

36. $\dfrac{a^2+b^2+c^2}{ab+bc+ca}$ 37. $(a+b+c)$.

38. $3(3\{3(3x-2)-2\}-2)-3=0$
$(3\{3(3x-2)-2\}-3=0$, $\therefore$ $3(3x-2)-3=0$, $x=1$.

39. 1. 40. 1. 41. 1. 42. 15.

43. $16\frac{1}{4}$. 44. 6. 45. 5. 46. $\dfrac{npqra+pqrb+qrc+rd}{mnpqr}$.

47. $-\frac{1}{6}$. 48. 0. 49. $-\frac{25}{136}$. 50. 1.

Exercise l., page 142.

1. 2, 3. 2. $\frac{1}{2}$, $\frac{1}{3}$. 3. ± 2, $1\frac{1}{4}$. 4. 1, $1\frac{1}{2}$.

5. $\pm\frac{2}{3}$, $\pm(a+b)$, a. 6. 4, 5; 2, $2\frac{1}{2}$.

7. -3 or 2; 4, -3; $2\frac{1}{3}$, $-1\frac{1}{3}$. 8. 1; $\frac{2}{3}$ or $\frac{3}{2}$; $\frac{1}{3}$ or 3.

9. $-\frac{2}{3}$ or $\frac{3}{2}$; $\frac{1}{6}$ or 6; $\frac{4}{3}$ or $-\frac{3}{2}$. 10. -1, 2, $-\frac{1}{2}$, 1.

11. 0, $-b$, $3b$. 12. a, $\pm a\sqrt{-1}$.

13. 1, $\dfrac{-1-\sqrt{5}}{2}$, $\dfrac{-1+\sqrt{5}}{2}$. 14. $\pm a$. 15. $\pm bc$, $-(b+c)$.

16. Factors are $(x-a)(x-a-2b)$, $\therefore$ $x=a+2b$.

17. b or $\pm a$. 18. $x+2ab$ is a factor, $\therefore$ $x=-2ab$. The other factor is $3x^2-2abx-2a^2b^2$, which gives $x=\frac{1}{3}ab(1\pm\sqrt{7})$.

19. $a-b$ is a factor, $\therefore$ also $x-a$, $b-x$ are factors; one linear factor remains which must be symmetrical in a, b, x, and is $\therefore$ $x+a+b$; $\therefore$ $x=a$, b, $-(a+b)$.

20. Transpose 1 to left-hand side, which then vanishes if $x-a=0$, or $x-b=0$; $\therefore$ $x=a$, or b.

21. Left hand $=\dfrac{(x-a)(x+a)^3}{(x+a)^3}=x-a$, which $=x^2-a^2$, $\therefore$ $x=a$, or $1-a$.

22. $=(x+a)(a+b)(x+b-a+b)=0$, &c.

23. $\therefore$ $ab(a-b)+bx(b-x)+ax(x-a)-(x-a)(b-x)=0$,
$\therefore$ $x-a=0$, $b-x=0$, $\therefore$ $x=b(1-b)\div(1-b+a)$.

24. $(x-5)(x+6)(x-7)=x^3-6x^2-37x+210.$

25. $(x-a)(x-4a)(x-3a)(x+4a)=$
$x^4-4ax^3-13a^2x^2+64a^3x-48a^4.$

26. $x(x-1)(x+2)(x-4)=0.$

27. $(x^2-2x-1)(x^2-2x-2)=x^4-4x^3+x^2+6x+2=0.$

Exercise li., page 146.

ANSWERS.

1. 4. 2. $-\frac{55}{7}$. 3. -107. 4. 8. 5. $3a$. 6. $-\frac{31}{159}$.
7. $5\frac{30}{59}$, 17. 8. 22, $46\frac{1}{3}$. 9. 7, 3. 10. 10, 10, 11.
11. 0 or 11; 33. 12. $3956\div3971$. 13. $\frac{1}{7}(15\pm\sqrt{190})$. 14. 3.
15. 3. 16. 4. 17. $22\div16$. 18 $1\frac{1}{2}$. 19. $3\frac{1}{4}$. 20. 4.
21. ±3. 22. 11. 23. 2 and $-1\pm\sqrt{-3}$. 24. $2\frac{1}{2}$. 25. 0.
26. $3a$. 27. $\frac{2}{3}$. 28. $\frac{16}{15}$. 29. 3. 30. 10.
31. 0, 1, or $(-5\pm\sqrt{-23})\div8$. 32 $102\frac{3}{5}$.
33. $(-11\pm\sqrt{4681})\div20$. 34. 2, $\frac{1}{3}$, or $\frac{7}{4}$. 35. -4.
36. 0, or $\pm\sqrt{(a^2+b^2)}$.

Hints and Solutions.

1. Combine first and last fractions $\therefore \dfrac{12x+2}{13x-16}=\dfrac{25}{18}$, &c.

2. As in last Example $\dfrac{9x+15}{5x-25}=\dfrac{13}{15}$, &c.

3. Completing divisions $\therefore \dfrac{72}{x-1}=\dfrac{70}{x+2}$, &c.

4. $\therefore \dfrac{11}{2x-9}=\dfrac{33}{21}$, &c.

5. Multiply through by 6, and complete the divisions,

$$\therefore \frac{6a}{x-a}-\frac{12a}{x+a}=0, \text{ \&c.}$$

6. $\therefore \dfrac{-29}{6x+5} = \dfrac{24}{6x-2} \therefore 84 = -53(3x-1)$, &c.

7. Multiplying by 2 and completing divisions, $\therefore$

$$\frac{47}{2x-15} + \frac{6}{x-5} = 0, \text{ \&c. } \frac{3}{x-8} = \frac{4}{x-5}, \text{ \&c.}$$

8. $\dfrac{8}{x-12} = \dfrac{12}{x-7}$; &c. $\dfrac{20}{x-13} - \dfrac{22}{x+7} = 0.$

9. $\therefore -\dfrac{15}{2x+1} + \dfrac{4}{x-3} = 0$, &c.; $\dfrac{-5}{x+2} + \dfrac{28}{5x+13} = 0.$

10. $\dfrac{7}{2x+1} = \dfrac{5}{2x-5}$; $\dfrac{3}{x-7} - \dfrac{3}{x-8} = \dfrac{1}{x-8} - \dfrac{1}{x-9}$

$\therefore \dfrac{3}{x-7} = \dfrac{1}{x-9} = 1$, by subtraction.

11. $\dfrac{17}{x-17} = \dfrac{15}{x-18} = \dfrac{3}{2}$, &c. $\dfrac{15}{2x+5} = 1 - \dfrac{4}{x-2}$, &c.

12. $\dfrac{5x-6}{28(x-1)} = \dfrac{289}{30}$, &c.

13. $\therefore \dfrac{x-3}{x^2-5x+6} - \dfrac{3x-2}{x^2-5x+6} = \dfrac{5}{x-5} \therefore -$

$$\frac{2x+5}{x^2-5x+6} = \frac{5x}{x^2-5x} = \frac{7x+5}{6}, \text{ \&c.}$$

14. $\dfrac{24}{x+1} + \dfrac{36}{x+3} = \dfrac{48}{x+1} \therefore \dfrac{3}{x+3} = \dfrac{2}{x+1} = \tfrac{1}{2}$, &c.

15. $\therefore \dfrac{1}{5x-4} = \dfrac{1}{7x-10}$. (By the usual divisions).

16. $-\dfrac{5}{x^2-9x+14} = -\dfrac{5}{x^2-7x+6}$, $\therefore$ &c.

17. $x+2+\dfrac{1}{x-5}+x+2+\dfrac{1}{x-9} = 2(x+2)+\dfrac{2}{x-8}$, &c.

18. $\dfrac{8x+12}{16x^2+48x+35} = \dfrac{8x+12}{16x^2+48x+32}$, $\therefore$ &c.

19. $\therefore \dfrac{1}{2x-4} + \dfrac{1}{2x-9} = \dfrac{1}{2x-8} + \dfrac{1}{2x-5}$,

or, $\dfrac{4x-13}{4x^2-26x+36} = \dfrac{4x-13}{4x^2-26x+40}$, $\therefore 4x-13=0$, &c.

20. $\therefore \dfrac{x}{4} + \dfrac{3}{14} - \dfrac{2x+4\frac{2}{7}}{23x-6} + \dfrac{x}{4} = \dfrac{11x}{21} - \dfrac{x}{42} + \dfrac{1}{14}$,

$\therefore \dfrac{2x+4\frac{2}{7}}{23x-6} = \frac{1}{7}$, &c.

21. $\therefore \dfrac{1}{x^2-6} + \dfrac{1}{x^2-12} = \dfrac{1}{x^2-8} + \dfrac{1}{x^2-10}$,

or $\dfrac{2x^2-18}{x^4-18x^2+80} = \dfrac{2x^2-18}{x^4-18x^2+8}$, $\therefore 2x^2-18=0$, &c.

22. $\therefore \dfrac{x}{2} - \dfrac{51}{52} - \dfrac{8}{52} + \dfrac{6x}{13} = x - \dfrac{5\frac{3}{4}x}{39} + \dfrac{2\frac{1}{2}}{39} - \dfrac{59}{52}$

$= \dfrac{x}{26} - \dfrac{7\frac{2}{3}x}{52} + \dfrac{3\frac{1}{3}}{52}$, $\therefore$ &c.

23. Multiply through by 18.

$\therefore \dfrac{6(1-2x)}{x^2-x+1} + \dfrac{9(1+x)}{x^2+1} + \dfrac{3}{x+1} = \dfrac{2}{x^2+1}$

$\therefore \dfrac{-6(2x-1)(x+1)+3(x^2-x+1)}{x^3+1} = -\dfrac{9x+7}{x^2+1}$, $\therefore 2x^3=16$.

24. Completing divisions as usual,

$\therefore \dfrac{-2}{2x-7} + \dfrac{1}{x-1} = -\dfrac{1}{x-4} + \dfrac{2}{2x-3}$

$\therefore \dfrac{-5}{(2x-7)(x-1)} = \dfrac{-5}{(x-4)(2x-3)}$, &c.

25. $\therefore 1 - \dfrac{a-b}{x-b} + 1 + \dfrac{a-b}{x-a} - \dfrac{(a-b^2)}{(x-a)(x-b)} = \dfrac{2(a-x)}{a+x}$

or $2+\dfrac{(a-b)^2}{(x-a)(x-b)} - \dfrac{(a-b)^2}{(x-a)(x-b)} = \dfrac{2(a-x)}{a+x)}$

$\therefore 2(a+x)=2(a-x)$, &c.

26. $\therefore \dfrac{6a}{3x+a} - \dfrac{9a}{2x+9a} = 0$, &c.

27. $\therefore \dfrac{1}{13\frac{1}{2}x-6} + \dfrac{1}{13\frac{1}{2}x-12} = \dfrac{1}{13\frac{1}{2}x-8} + \dfrac{1}{13\frac{1}{2}x-10}$

or $\dfrac{27x-18}{\cdots\cdots} = \dfrac{27x-18}{\cdots\cdots} \therefore 27x-18=0.$

28. Sum of first two fractions is $\dfrac{x}{2(x-1)^2} \therefore x$ is a factor, $\therefore$

$\dfrac{1}{2(x-1)^2} - \dfrac{1}{2(x^2+1)} = \dfrac{16}{(x-1)(x^2+1)}$, &c.

29. $\dfrac{x}{3} + 2 - 2\frac{1}{2} + \dfrac{x}{3} = 3 - \dfrac{x}{2}$, &c.

30. Multiply by $\frac{2}{3}$ and factor, $\therefore x - \dfrac{6(3x-1)(3x+1)}{(5x-1)(x+3)} = 2x -$

$\dfrac{2x^2-1}{x+3} - 19 + x$; perform divisions $\therefore \dfrac{65}{x+3} = 5$, &c.

31. Perform divisions, $\therefore \dfrac{3}{2(x-1)} - \dfrac{1}{2(x+1)} = \dfrac{5x+1}{(x-1)^2}$

$\therefore \dfrac{x+2}{x^2-1} = \dfrac{5x+1}{(x-1)^2}$, or $\dfrac{x+2}{x-1} = \dfrac{5x+1}{x-1} \therefore x-1=0.$

32. $\dfrac{2x}{3} - \dfrac{20}{7} - 10 - \dfrac{23}{\frac{1}{2}x-3} - \dfrac{2}{21} + x = \dfrac{2x}{3} - \dfrac{19}{7} - \dfrac{1}{7} - 10$

$-\dfrac{4}{7} + x$, or $\dfrac{23}{\frac{1}{2}x-3} = \dfrac{10}{21}.$

33. $\therefore \dfrac{18x-22}{13-2x} = \dfrac{1}{2} + \dfrac{5x}{4} = \dfrac{2+5x}{4}$, &c.

34. $-\dfrac{1}{3x-1} - \dfrac{16}{4x-7} = -\dfrac{50x-19}{12x^2-25x+7}$,

or $\dfrac{52x-23}{(3x-1)(4x-7)} = \dfrac{50x-19}{(3x-1)(4x-7)} \therefore 3x-1=0,$

$4x-7=0$, &c,

35. $\frac{5}{2x+5}+\frac{5}{2x+11}=\frac{5}{2x+9}+\frac{5}{2x+7}\ \therefore\ \frac{4x+16}{4x^2+32x+55}$

$\therefore\ \frac{4x+5}{4x^2+32x+63}\ \therefore\ 4x+16=0$, &c.

36. Sum of first and last fractions $=\frac{2x}{x^2-(a+b^2)}$; sum of second and third $=\frac{2x}{x^2-(a-b)^2}$, $\therefore 2x\left\{\frac{1}{x^2-(a+b)^2}+\frac{1}{x^2-(a-b)^2}\right\}$ $=0$, $\therefore\ x=0$, or $2x^2=(a+b)^2+(a-b)^2=2(a^2+b^2)$, &c.

Exercise lii., page 150.

1. (*a*) $\therefore\ \frac{2x}{2}=\frac{1-a}{1+a}$. (*b*) $\therefore\ \frac{2x}{2a}=\frac{m+1}{m-1}$.

(*c*) $\therefore\ \frac{2ax}{2b}=\frac{m+n}{m-n}$.

2. (*a*) $(a-b)$. (*b*) 0. (*c*) $\frac{a}{a-x}=\frac{b}{b-x}=\frac{a-b}{a-b}=1$

$\therefore\ x=0$. 3. b; $\frac{ma}{b}$; $\frac{b}{ca}$.

4. (*a*) $\therefore\ (x-1)\left(\frac{1}{a+b}-\frac{1}{c+d}\right)=0$, $\therefore\ x=1$.

(*b*) -1. (*c*) $\frac{a-x}{b-x}=\frac{a+x}{b+x}=\frac{a}{b}$. $\therefore\ x=0$.

5. $\therefore\ \frac{2x+3}{2x^2-7x+3}=\frac{2x+3}{x^2-9x+2}$, $\therefore\ x=-\frac{3}{2}$, or -1.

6. $\therefore\ \frac{2ax}{2(b-c)}=\frac{(b-c)^2+(b+c)^2}{(b-c)^2-(b+c)^2}$, and $x=\frac{(c-b)(b^2+c^2)}{2abc}$.

7. Add and subtract, $\therefore\ \frac{2\sqrt{(x+y)}}{2\sqrt{(x-y)}}=\frac{x+y}{x-y}$; divide each side by $\frac{\sqrt{(x+y)}}{\sqrt{(x-y)}}$ $\therefore\ \frac{x+y}{x-y}=1$.

8. (*a*) Subtract numerators from denominators.

$$\therefore \frac{2x-7}{4} = \frac{x+7}{4} \quad \therefore x=14.$$

(*b*) Subtract numerators from twice denominators.

$$\therefore \frac{4x-5}{15} = \frac{10x-32}{15} \qquad \therefore 6x=27,\ x=4\tfrac{1}{2}.$$

9. (*a*) Each fraction = difference of numerators divided by difference of denominators $=\frac{82}{82}=1$. $\therefore x=2$.

(*b*) Same as last: each $=\frac{58}{-58}=-1$. $\therefore x=\frac{6}{295}$.

10. (*a*) $$\frac{210x-73}{310x-66} = \frac{210x-73}{310x+80} \quad \therefore 210x=73.$$

(*b*) $$\frac{mx-a-b}{nx-c-d} = \frac{mx-a-c}{nx-b-d} = \frac{c-b}{b-c} = -1,$$

$$\therefore x=(a+b+c+d)\div(m+n).$$

11. Take difference of numerators divided by difference of denominators, and each fraction $=1$, $\therefore x = \dfrac{b}{a}$.

12. $$\frac{x^3+ax^2-bx+c}{x^3-ax^2+bx+c} = \frac{x^3+ax^2-bx}{x^3-ax^2+bx} = \frac{c}{c} = 1$$

$$\therefore x^2+ax-b=x^2-ax+b, \text{ or } 2ax=2b,\ x=\frac{b}{a}.$$

13. $$\frac{b\sqrt{(2a-x)}}{\sqrt{(2a^2-x^2)}} = \frac{b}{\sqrt{a}} \quad \therefore \frac{2a-x}{2a^2-x^2} = \frac{1}{a} = \frac{2a}{2a^2}$$

$$= \frac{x}{x^2} = \frac{1}{x} \quad \therefore x=a, \text{ or } 0.$$

14. $$\frac{\sqrt{(x^2+a^2)}}{\sqrt{(x^2-a^2)}} = \frac{a^2+1}{a^2-1} \quad \therefore \frac{x^2}{a^2} = \frac{(a^2+1)^2+(a^2-1)^2}{(a^2+1)^2-(a^2-1)^2}$$

$$= \frac{a^4+1}{2a^2} \quad \therefore x=\pm\sqrt{\left(\frac{a^2+1}{2}\right)}.$$

15. Solve same as 12th, $\therefore 12x=8,\ x=\frac{2}{3}$.

16. $\therefore \dfrac{\sqrt[3]{(x+1)}}{\sqrt[3]{(x-1)}}=3, \quad \dfrac{x+1}{x-1}=27, \therefore x = \dfrac{14}{13}.$

17. $\therefore \dfrac{\sqrt{x}}{28} = \dfrac{\sqrt{x}}{18+5\sqrt{x}} \therefore x=0$, or $5\sqrt{x}=10$, &c.

18. Solve same as in 12th, and $x=c\div ab$.

19 $\dfrac{9\sqrt{(2x-1)}}{\sqrt{(2x-1)}+4\sqrt{(3x-3)}} = \dfrac{25}{12} \therefore \dfrac{\sqrt{(2x-1)}}{\sqrt{(3x-3)}} = \dfrac{100}{83}$, &c.

20. $\therefore \dfrac{\sqrt{2x}}{\sqrt{(3-2x)}} = \dfrac{5}{1} \therefore x=\dfrac{75}{52}.$

21. $\therefore \dfrac{2\sqrt[3]{(3x+3)}}{\sqrt[3]{(7x+8)}} = \dfrac{6}{4}, \ \dfrac{7x+8}{x+1}=7\tfrac{1}{9} \therefore \dfrac{1}{x+1} = \dfrac{1}{9},$

$\therefore x=8.$

22. $\dfrac{13+2\sqrt{(x-5)}}{13-2\sqrt{(x-5)}} = 11 \therefore \dfrac{2\sqrt{(x-5)}}{13} = \dfrac{5}{6} \therefore x=34\tfrac{49}{144}.$

23. $\dfrac{\sqrt{(nx+1)}+\sqrt{(nx)}}{\sqrt{(nx+1)}-\sqrt{nx}} = \dfrac{\sqrt{n}+1}{\sqrt{n}-1} \therefore \dfrac{\sqrt{(nx+1)}}{\sqrt{nx}}=\sqrt{n}.$

$\dfrac{nx^2+1}{nx} = \dfrac{n}{1} \therefore x=1\div n(n-1).$

24. $\dfrac{\sqrt{(x+c)}}{\sqrt{b}} = \dfrac{\sqrt{x}}{\sqrt{a}} \therefore x = \dfrac{ac}{b-a}.$

25. (a). Each fraction = difference of numerators divided by difference of denominators $=5$, $\therefore x=4$.

(b). $\dfrac{2}{\sqrt[3]{2x}+9}=\dfrac{3}{\sqrt[3]{2x}+15}=\dfrac{1}{6} \therefore x = 3\tfrac{3}{4}$, or $x=13\tfrac{1}{2}$.

26. (a). Each fraction $= \dfrac{a}{b} = \dfrac{\sqrt{x}+2a}{\sqrt{x}+b} = \dfrac{\sqrt{x}+a}{\sqrt{x}}$

$\therefore x = a^2b^2\div(a-b)^2$. ($b$). $\dfrac{\sqrt{3x}-1}{1}=\dfrac{\sqrt{3x}+1}{2}=2 \therefore x=3.$

27. $\dfrac{x}{a} = \dfrac{4a}{(1+a)^2}; \ \sqrt{\dfrac{x}{b}} = \dfrac{a+b}{a-b}$ &c

28. Divide numerator and denominator by $\sqrt{(ax+1)}$, then as usual $\dfrac{\sqrt{(ax)+1)}}{\sqrt{(ax)-1)}} = \dfrac{b+1}{b-1}$, $x=(b^2+1)\div 2ab$, &c.

29. As in 28, $1 \div \sqrt{\{1-\sqrt{(1-x)}\}} = (1+a)\div(1-a)$, &c.

30. Square both sides, then $a^2 \div (x^2+2ax) = 4b \div (b-1)^2$, &c.

31. Proceeding by addition and subtraction we find

$(x+1)^5 \div (x-1)^5 = (a^5+5a^4+20a^2+2)\div(5a^4-a^5)$, or $\dfrac{x+1}{x-1} = $ &c

Exercise liii., page 153.

1. We have the *identity* $x+4-(x-3)=7$, divide this, member by member, by given equation, $\therefore x=8$.

2. Transpose and square $x=0$, which satisfies the equation

3. $x=3$. 4 $\sqrt{x}(\sqrt{m}-\sqrt{n})=m-n$, $\therefore x=(\sqrt{m}+\sqrt{n})^2$.

5. $\sqrt{bx}-\sqrt{x}=-\sqrt{(ab+bx)}$ square, &c., $x=ab\div(1-2\sqrt{b})$.

6. $x=\dfrac{4}{7}$. 7. $x=\dfrac{1}{a-2}$. 8. $x=\dfrac{18962}{12393}$. 9. $\dfrac{\sqrt{a}}{\sqrt{a+2}}$.

10. $(b+x)^2-(b^2+x^2)=2bx$ *identically*, divide this, member by member, as in 1 above, and add, $2(b+x)=$

$$\frac{2bx}{c^2}+c^2 \quad \therefore x=\frac{c^4-2bc^2}{2c^2-2b}.$$

11. $x=\frac{1}{3}$. 12. $2x-(2x-27a)=27a$ identically, then proceed as in example 1 above $x=18a$.

13. Cube by formula [VI] $1-x+1+x+3\sqrt[3]{(1-x^2)}\times\sqrt[3]{3}$ (by substituting cube root of 3 for its value) $=3$, $\therefore x^2=\frac{80}{81}$.

14. Cube and substitute as in last, $\therefore 3\sqrt[3]{7}\sqrt[3]{(9-x^2)}=1$, $\therefore x=\pm\frac{10}{3}\sqrt{\frac{17}{21}}$.

15. Proceed as in last two questions, $\therefore x=\pm\dfrac{4}{11}\sqrt{-11}$.

16. $x=\pm\sqrt{\left\{a^2-\dfrac{(b-2a)^3}{27b}\right\}}$. 17. $x=0$. 18. $x=\frac{9}{16}a$.

19. Proceed as in example 3, page 152, $(c-a-b)^3=27abc$.

20. $\therefore x\sqrt{(a^2+x^2)}=(n-1)a^2-x^2$, and $x^2(2n-1)=(n-1)^2a^2$, &c. 21. $16xy=(n-4x-y)^2$.

22. $1+x-\{1+x+\sqrt{(1-x)}\}=\sqrt{(1-x)}$ identically; divide member by member, and add given equation, $\therefore 2\sqrt{(1+x)}=\sqrt{(1-x)}-1$, &c., $x=0$, $-\frac{24}{25}$.

23. $\sqrt[4]{x}$ is a factor, $\therefore x=0$; dividing, $\therefore \sqrt{(\sqrt{x}+1)}-\sqrt{(\sqrt{x}-1)}=a\div\sqrt{(\sqrt{x}+1)}$, and $\sqrt{x}+1-(\sqrt{x}-1)=2$ *identically*,

$$\therefore \sqrt{(\sqrt{x}+1)}+\sqrt{(\sqrt{x}-1)}=\frac{2\sqrt{(\sqrt{x}+1)}}{a}\ \therefore 2\sqrt{(\sqrt{x}+1)}=$$

$$\frac{2\sqrt{(\sqrt{x}+1)}}{a}+\frac{a}{\sqrt{(\sqrt{x}+1)}}\ \therefore (2a-2)\sqrt{(\sqrt{x}+1)}=$$

$$a^2\div\sqrt{(\sqrt{x}+1)}, \text{ \&c., } x=\left(\frac{a^2}{2a-2}-1\right)^2.$$

24. $1+x+x^2-(1-x+x^2)=2x$, identically, divide, &c.,

$$2\sqrt{(1+x+x^2)}=\frac{2}{m}+mx;\ x=\frac{2}{m}\sqrt{\frac{1-m^2}{4-m^2}}.$$

25. $a^2-x^2-x^2(a^2-1)=a^2(1-x^2)$, identically divide, &c., $\therefore$ $2\sqrt{(a^2-x^2)}-x\sqrt{(a^2-1)}=1$;

$$\therefore x=\frac{a^2-1}{a^2}\{a^2+2\pm\sqrt{(a^2+1)}\}.$$

26. Reduce first fraction, $\therefore \sqrt{bx+c}=\dfrac{\sqrt{bx+c}}{n}-a$,

$$\therefore x=(cn-an+c)^2\div b(n-1)^2.$$

27. $2x^2+5-(2x^2-5)=10$ identically, then as in former examples. $2\sqrt{(2x^2+5)}=2\sqrt{15}$, $x=\pm 5$.

28. $3x^2+10-(3x^2-10)=20$ identically, $\therefore 2\sqrt{(3x^2+10)}=\frac{17}{7}\sqrt{17}-\frac{3}{7}\sqrt{3}$, &c.

29. $3x^2+9=(3x^2-9)=18$ identically, &c., $x=\pm 5$.

30. $3a-3b+x^2-(2a-2b+x^2)=a-b$ identically, &c. $x=\pm\sqrt{(3b-2a)}$.

31. $4a^2-3b^2-2x^2-(3a^2-3b^2-x^2)=a^2-x^2$ identically, &c. $x=\sqrt{\frac{3}{2}(a^2-b^2)}$.

32. Cube and substitute as in example 21 above, $(2y+2z-2x)^3+216xyz=0$.

33. Eqnation is $\sqrt{(a+x)}+\sqrt{(a-x)}=2x\div\sqrt{\{a+\sqrt{(a^2+x^2)}\}}$; we have identically $a+x-(a-x)=2x$, divide, &c., $x=\frac{2}{3}a\sqrt{6}$.

34. Remove the factor $\sqrt{x}$, and reduce the fraction, $\therefore\ x=0$. Also, $x+2a-(x-2a)=4a$ identically, divide as in former cases and clear of fractions, $\therefore\ 2(x+2a($=

$$\frac{4}{n}(x+2a)+na\,;\ x=a\left(\frac{n^2-4n+8}{2n-4}\right).$$

35. $x=a^2+2a$

36. $(2a+x)^2+b^2-\{(2a-x)^2+b^2\}=8ax$ identically, $\therefore$ $2\sqrt{\{(2a+x)^2+b^2\}}=2a+4x$, $x=\pm\sqrt{(3a^2+b^2)}\div\sqrt{3}$.

Exercise liv., page 157.

1. Right-hand member is $(x^2-ax+a^2)(x-b)$; divide by common factor and clear of fractions, $x^2+ax+a^2=x^2-b^2$, $\therefore\ x=$&c.

2. Right-hand member $=(x-b)(x^2+2ax+2a^2)$
$\therefore\ x^2+2a^2-2ax=x^2-b^2$, &c.

3. $x=\dfrac{(a-b)a^2-2c(a^2+ab+b^2)}{a^2-2c(a^3-b^3)}$.

4. Right-hand member is $2ab(x+b)x^2$; left-hand member factors into $\dfrac{-(x+a)(x+b)(a+b)}{abx(x+a+b)}$ $\therefore\ x+b$ is a factor, &c.

5. $$\frac{1}{x-b}\left\{\frac{1}{x-c}-\frac{1}{a+b}\right\}=\frac{1}{a+c}\left(\frac{1}{x-c}-\frac{1}{a+b}\right)$$

$$\therefore\ \frac{1}{x-b}=\frac{1}{a+c}\ \therefore\ x=a+b+c.$$

6. $\frac{ab}{a-b}\left\{\frac{ab}{(a-b)^2}-3\right\}=x\left\{3-\frac{b}{a}\left(1+\frac{b(2a-b)}{(a-b)^2}\right)\right\}=$

$x\left\{3-\frac{b}{a}\left(\frac{a}{a-b}\right)^2\right\}=x\left\{3-\frac{a}{(a-b)^2}\right\}$

$\therefore x=ab\div(b-a)$.

7. Denominator of left-hand member $=(x^2-3ax-a^2)$ $(x^2+3ax-a^2)$; invert both members, $\therefore x^2-3ax-a^2$ is a factor giving $x=\frac{1}{2}a\{3a\pm\sqrt{13}\}$; result of division is $x^2-(2a+1)x=-\frac{a^2}{2}$, &c.

8. Equation may be arranged, multiplying by 2, thus:

$\left(\frac{x-a}{x+a}\right)^2-2.\ \frac{x-a}{x+a}+x-a=0$, where $x-a$ is a factor, &c.

9. Right hand member $=x^3-(a+b+c))x^2+(ab+bc+ca)x-abc$

$\therefore (a^2+b^2+c^2-ab-bc-ca)x=\frac{1}{3}(a^3+b^3+c^3-3abc)$

$\therefore x=\frac{1}{3}(a+b+c)$.

10. Transpose negative term in right-hand member and combine the fractions $\therefore \frac{2bc+2ca+2ac+a^2+b^2+c^2}{2abcx}=\frac{(a+b+c)^2}{2}$

$\therefore \frac{1}{abcx}=1,\ x=1\div abc$.

11. Multiply through by abc, then $a+b+c-(a^2+b^2+c^2)x=$ $(2ab+2ac+2ca)x,\ \therefore\ (a+b+c)^2x=a+b+c$.

12. Transpose and combine terms involving $\frac{a-b}{abc}$; $x=$

$(a-b)(ac-2b)\div(a+b)ac$.

13. $x^3+(b+c)^3+3b(b+c)x-b^3=0,\ \therefore\ x+b+c-b$ is a factor.

14. As in last example $\sqrt[3]{x}-\sqrt[3]{a}-\sqrt[3]{b}$ is a factor.

$\therefore x=(\sqrt[3]{a}+\sqrt[3]{b})^3$.

15. $11(x^4-16)+10x(x^2-4)=0,\ \therefore\ x^2-4=0$, &c.

16. $\frac{a}{(a-b)^2}\left\{x-\frac{c}{a-b}\right\}=\frac{1}{(a+b)^2}\left\{x-\frac{c}{a-b}\right\}\therefore x=\frac{c}{a-b}$

17. The equation can be put in the form—

$$-\frac{a-b}{a+b}\left(x^2-\frac{1-cx}{1+cx}\right)+x^3+\frac{2cx^2}{1+cx}-x=0,$$

$$\text{or } -\frac{a-b}{a+b}\left(x^2-\frac{1-cx}{1+cx}\right)+x\left(x^2-\frac{1-cx}{1+cx}\right)=0 \therefore x-\frac{a-b}{a+b}$$

is a factor, &c.

18. $$\frac{(2x^2-2a^2+5ax)(2x^2-2a^2-5ax)}{2x+a}$$

$$=\tfrac{1}{2}(2x+a)(2x^2-2a^2-5ax) \therefore 2x^2-2a^2-5ax=0,$$

or $2x^2-2a^2+5ax=\frac{1}{2}(x+a)^2$, $x=\frac{5}{6}a$.

19. $$\frac{7}{x-7}\left(\frac{1}{x-4}+\frac{1}{x-10}\right)=\frac{7x^2}{2(x-4)(x-10)} \text{ or } \frac{14}{(x-4)(x-10)}$$

$=7x^2\div2(x-4)(x-10)$, $\therefore x^2=4$.

20. Proceeding as in last example,

$$\frac{8(2x-10)}{(x-1)(x-5)(x-9)}=\frac{x4}{(x-1)(x-9)} \text{ or } x^4=16, x^4-16=0, \text{ \&c.}$$

21. Add, term by term, the identity used in example 3, page 155, then $\dfrac{2x}{(a-b)(c-a)}+\dfrac{2x}{(b-c)(c-a)}=\dfrac{a+c}{(a-b)(b-c)(c-a)}$

$\therefore$ $2x(b-c+a-b)=$&c., $x=\frac{1}{2}\left(\dfrac{a+c}{a-c}\right)$.

22. As in example 4, page 156, left-hand member $=$

$3(x-a)(a-b)(b-x)=x^2-a^2$, $x-a$ is a factor :

Also $x=(3ab-3b^2-a)\div(1+3a-3b)$.

23. Left-hand member is $\dfrac{(x-a)^3(x+a)}{(x-a)^3}$ which $=2a$ $\therefore x=a$.

24. First member vanishes for $x+a=0$, $a+b=0$, and $x-b=0$, and becomes $3(x+a)(a+b)(x-b)=$&c.

25. Left-hand member vanishes for $b=0$, $x-a=0$, and $a-b=0$; numerical coefficient is found to be 6, $\therefore 6b(x-a)(a-b)=(a-b)c^2$, $x=(c^2+6ab)\div6b$.

26. First member vanishes for $a-b=0$, $a+b=0$, $x-a=0$, and numerical coefficient $=6$, $\therefore$

$$6(a-b)(a+b)(x-a)=(a^2-b^2)c, \ \therefore\ x=\tfrac{1}{6}(c+6a).$$

27. Clear of denominators, $x\{x^3+a^3-(x^3-a^3)=a^4$, $\therefore\ x=\frac{1}{2}a$.

28. First number vanishes if $x=0$, $a=0$, or $b=0$, $\therefore\ abx$ is a factor; one linear factor remains which must be symmetrical in a, b, x, and is $\therefore\ x+a+b$; numerical factor is found to be 12, $\therefore$ $12abx(x+a+b)=12ab\{x^2+(a+b)^2\}$ $\therefore\ x=a+b$.

29. Arrange thus: $\dfrac{a}{a^2-bc}+\dfrac{b}{b^2-ca}+\dfrac{c}{c^2-ab}=$

$$x\left\{\left(\frac{1}{a^2-bc}+\frac{1}{ab+bc+ca}\right)+\text{anal.}+\text{anal.}\right\}=$$

$$x\left\{\frac{a^2+ab+ca}{(a^2-bc)(ab+bc+ca)}+\text{anal.}+\text{anal.}\right\}=$$

$\dfrac{x(a+b+c)}{ab+bc+ca}\left\{\dfrac{a}{a^2-bc}+\dfrac{b}{b^2-ca}+\dfrac{c}{c^2-ab}\right\}$, where the quantity within the $\{\ \}$ is common to both members of the equation; strike this out, and $x=(ab+bc+ca)\div(a+b+c)$.

30. First member vanishes if $b-a^2=0$. and by symmetry, if $x-b^2=0$, or $a-x^2=0$, and the *numerical* factor is found to be -1, $\therefore$ $-(b-a^2)(x-b^2)(a-x^2)=(a-x^2)(b-a^2)(b+x^2)$, $\therefore$ $a-x^2=0$, &c.

31. $1+x+x^2$ is a factor, and $\therefore\ x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$; also from the quotient we get $\dfrac{1+x+x^2}{1-x+x^2}=\dfrac{ax+1}{ax-1}$, add and subtract $\therefore$ $\dfrac{1+x^2}{x}=ax$, &c.

32. Transpose, square and perform division in left-hand member then $\dfrac{a-b}{x+b}=\dfrac{1}{4}\left(\dfrac{a-b}{x+c}\right)^2+\dfrac{a-b}{x+c}$ divide by $a-b$ $\therefore$ $\dfrac{1}{x+b}-\dfrac{1}{x+c}$ $=\dfrac{a-b}{4(x+c)^2}$, or $\dfrac{c-b}{x+b}=\dfrac{a-b}{4(x+c)}$, &c.

33. Complete the divisions, square and transpose,

$$\frac{1}{x+b} - \frac{2}{2x+b+c} = \frac{a-b}{(2x+b+c)^2}, \text{ or } \frac{c-b}{x+b} = \frac{a-b}{2x+b+c}$$

$\therefore\ x = (c^2 - ab) \div (a + b - 2c)$.

34. Factor, then $\sqrt{\{(x+12)(x+15)\}} - \sqrt{\{(x+12)(x+14)\}} = \sqrt{\frac{x+15}{x+12}}\ \therefore\ (x+12)\sqrt{(x+15)} - (x+12)(x+14) =$

$\sqrt{(x+15)}$, $\therefore\ (x+11)\sqrt{(x+15)} = (x+12)\sqrt{(x+14)}$, square then $x^2+29x+201=0$, $x=\frac{1}{2}(-29 \pm \sqrt{37})$.

35. Put $x+a=m$, and $\sqrt{(x^2+2ax+2b^2)}=n$, then $(m+n)^3 + (m-n)^3 = 14m^3$ or $2m(m^2+3n^2)=14m^3$, $\therefore\ n^2=2m^2$ restore the values of m and n, and $2x^2+4ax+2a^2 = x^2+2ax+2b^2$, or $(x+a)^2 = 2b^2 - a^2$, &c.

36. Left-hand member is of the form $(x+y)^2 + (x-y)^2$

$\therefore 2(x+a)^2 + 2(x^2 - 2ax + 2b^2) = x^2 - b^2 + 2a(a-b)$,

$3x^2 = 3b^2 - 2ab$, $x =$ &c.

37. Put $x+a=m$, and $x-b=n$, $\therefore\ a+b=m-n$,

$\therefore \dfrac{m^3}{n^3} = \dfrac{2m-n}{2n-m}$ which gives $m+n=0$, or $2x+a-b=0$,

$\therefore\ x = \frac{1}{2}(b-a)$.

38. Factor first member, $\therefore\ 3(x-1)(39x^2-120x+93) = 27(x^3-1)$ $\therefore\ x-1=0$. Also, $13x^2-40x+3=3x^2+3x=3$, or $10x^2-43x+28=0$, or $(5x-4)(2x-7)=0$, $\therefore$ &c.

39. Proceed as in example 6, page 156, putting y for x^2-6x;

$$\frac{1}{y-4} + \frac{1}{y-9} - \frac{2}{y-16} = \frac{4}{y-9}$$

$\therefore\ 4y^2 - 61y = -120$, $y = \frac{1}{8}\{61 \pm \sqrt{1801}\} = a$ suppose $\therefore$ $x^2 - 6x = a$, &c.

40. As in last example, write y for x^2-2x and equation becomes on performing divisions, $\frac{1}{y-8}+\frac{1}{y-24}-\frac{2}{y-48}=0$,

or $\frac{1}{y-8}-\frac{1}{y-48}+\frac{1}{y-24}-\frac{1}{y-48}=0$, or $\frac{40}{y-8}=\frac{24}{y-24}$,

or $\frac{5}{y-8}=\frac{-3}{y-24}=\frac{8}{16}=\frac{1}{2}$ (by subtraction) $\therefore y=18$,

$\therefore x^2-2x=18,\ x=1\pm\sqrt{19}$.

41. Put $x+a-b=m$, and $\sqrt{(x^2+a^2-b^2)}=n$, then the equation becomes $(m+n)^3+(m-n)^3=8m^3$ which gives $m=0$, $\therefore$ $x+a-b=0$: also $m^2+3n^2=4m^2n^2=u^2$, *i. e.*, $(x+a-b)^2=x^2+a^2--b^2$, which gives $x=b$.

42. Factor denominators and transpose, then

$$\frac{1}{x+a+b}\left\{\frac{1}{x+a-b}-\frac{1}{x-a-b}\right\}=\frac{1}{x-a+b}\left\{\frac{1}{x+a-b}-\frac{1}{x+a+b}\right\}$$

or $-\frac{2a}{(x+a+b)(x+a-b)(x-a-b)}=\frac{-2b}{(x-a+b)(x+a-b)(x+a+b)}$

$$\therefore\ \frac{b}{x-a+b}=\frac{a}{a+b-x}\qquad\therefore x=\frac{a^2+b^2}{a+b}.$$

43. By division $41\left\{6+\frac{39}{x+1}+7+\frac{39}{x+4}\right\}+130=$

$$39\left\{8+\frac{41}{x+2}+9+\frac{41}{x+3}\right\}\therefore\frac{1}{x+1}+\frac{1}{x+4}=\frac{1}{x+2}+\frac{1}{x+3},$$

$\therefore$ $2x+5$ is a factor, &c.

44. Divide and proceed as in last, then

$$51\left\{16+\frac{61}{x-1}-9-\frac{61}{x-4}\right\}+863=61\left\{27-\frac{51}{x-2}-7-\frac{51}{x-3}\right\}$$

or $51\times7+863-61\times20+$

$$51\times61\times\left\{\frac{1}{x-1}-\frac{1}{x-4}\right\}=61\times51\left\{\frac{1}{x-2}-\frac{1}{x-3}\right\}$$

or $\frac{-3}{x^2-5x+4}=\frac{-1}{x^2-5x+6}=\frac{2}{2}=1$ by subtraction.

$\therefore$ $x^2-5x+4=-1,\ x=\frac{1}{2}(5\pm\sqrt{3})$.

45. Take first and fourth factors together, also second and third. $\therefore (x^2+7ax+6a^2)(x^2+7ax+12a^2)=$ &c., or $(x^2+7ax+6a^2)^2+6a^2(x^2+7ax+6a^2)=x^4+6a^2(x^2+7ax+6a^2)$,
$\therefore (x^2+7ax+6a^2)-x^4=0$, or $(7ax+6a^2)(2x^2+7ax+6a^2)=0$,
$\therefore x=\frac{6}{7}a$, also $=\frac{3}{2}a$, or $-2a$.

46. Transpose third fraction $\frac{1}{x+6a}+\frac{2}{x-3a}=\frac{6}{x+a}-\frac{3}{x+2a}$,

$$\therefore \frac{3x+9a}{\text{denominator.}}=\frac{3x+9a}{\text{denominator.}} \quad \therefore 3x+9a=, \text{ \&c.}$$

Exercise lv., page 160.

1. $bc\div(a+c)$. 2. $(a^2+b-2ab)\div(a+b^2)$. 3. $(ad-bc)\div(a-b)$. 4. a. 5. Equation is

$$\frac{(a-b)(a-c)}{(a+b)(a+c)}=\frac{x-a}{x+a} \therefore \text{by (6), page 122, } \frac{2a^2+2bc}{2(b+c)a}=\frac{x}{a} \therefore x=\frac{a^2+bc}{b+c}.$$

6. $(a-b)\{a+(x-c)\}+(a+b)\{a-(x-c)\}=2a^2$, $\therefore 2a^2-2b(x-c)=2a^2$, $\therefore x=c$; *or*, first member becomes $2a^2$ if $x-c=0$, $\therefore x=c$.

7. $(m+a)\{a+(b-x)\}+(a-m)(b-x)=a(m+b)$,
$\therefore 2a(b-x)=a(b-a)$, $\therefore x=\frac{1}{2}(a+b)$.

8. $m\{(a+b)-x\}=n\{x-(a+b)\}$, $\therefore (m+nx)=(m+n)(a+b)$,
$\therefore x=a+b$.

9. $m^2-n^2-(m+n)x+(m-n)x=m^2-n^2$, $\therefore 2mx=0$.

10. By division $x\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{p}\right)=0$, $\therefore x=0$.

11. Divide, and transpose, $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=ab+bc+ca$
$\therefore x=abc$.

12. As in last, $x(a+b+c)=a^2+b^2+c^2$, &c.

13. $x=(a+b+c)\div(a^2+b^2+c^2)$.

14. $x(ab+bc+ca)=a^2+b^2+c^2)$.

15. $\frac{1}{a+b}\left\{a^2+b^2+(a+b)c\right\}x=\frac{1}{a+b}\{a^2+b^2+(a+b)c\}$

$\therefore x=\frac{a+b}{a-b}.$

16. $\frac{ab}{a+b}\left\{3c+\frac{ab}{(a+b)^2}\right\}+\frac{x}{a}\left\{\frac{(2a+b)b^2}{(a+b)^2}-b-3ac\right\}=0.$

or $\frac{ab}{a+b}\left\{3c+\frac{ab}{(a+b)^2}\right\}-x\left\{\frac{ab}{(a+b)^2}+3c\right\}=0, \therefore x=\frac{ab}{a+b}.$

17. $\frac{1}{x}=\frac{2}{9}, x=4\frac{1}{2}.$ 18. $\frac{5}{12x}=\frac{1}{12} \therefore x=5.$

19. Equation reduces to $\frac{1}{x}=\frac{1}{4}, \therefore x=4.$

20. Transpose last fraction to left-hand member and combine the three fractions.

$$\therefore\ 6=7+\frac{119}{x+21}, \therefore x=-140.$$

21. $(10+3+7)\div 2(x+3)=\frac{1}{2}, \therefore x=17.$

22. Transpose second fraction of left-hand member to right-hand, and take the three fractions together.

$$\therefore \frac{6x+5}{8x-15}=\frac{15}{15}=1, \therefore x=10.$$

23. $\frac{ax-1}{ax+1}-\frac{1}{x}=\frac{ax-1}{ax+1}-\frac{1}{a}, \therefore x=a.$

24. $\frac{a^2dx-a^2e^2}{bdx-be^2-c^2x}=1, \therefore x=\frac{e^2(a^2-b)}{a^2d+c^2-bd}.$

25. $x=3\frac{1}{4}$; $x=0.$ 26. $x=3\frac{1}{21}.$

27. $\frac{ab}{a+b}$; as second equation stands it is a quadratic; right-hand member should be c^2-x^2, then $x=(ab-c^2)\div(a+b).$

28. $x=-b$; $x=a$. 29. $x=0$; $x=0$. 30. $x=\frac{1}{2}(a+b-c).$

31. $x=\frac{ab}{a+b}.$ 32. $x=d$. Equation becomes an identity when $x-d=0.$

33. $(a-b)(a+b)^2x = ab(a+b)$, $x = ab \div (a^2-b^2)$.

34. $\dfrac{x+3}{x-3} = \dfrac{x^2+9x+20}{x^2+3x+2}$, $\therefore$ by division $\dfrac{6}{x-3}$

$= (6x+18) \div (x^2+3x+2) \quad \therefore x = -3\frac{2}{3}$.

35. Apply formula B.

$\therefore x^3+6x^2+11x+6 = x^3-6x^2+11x-6 \quad \therefore x = \frac{3}{5}$.

36. As in last Ex., $x = -3\frac{2}{3}$.

37. Equation reduces to $50 = 54$, $\therefore$ there is no finite value of x.

38. Apply formula B, $x = 10$.

39. Apply formula B, $x = abc \div (ab+bc+ca)$

40. Apply formula B, $x = (ab+bc+ca-ad-bd-cd) \div (a+b+c-3d)$.

41. Apply formula B to obtain product of second term,

$\therefore x^3-2x^2a+xa^2-x^3+2x^2a-$
$\{(a-b)(a-c)+(a-c)(b+c)+(b+c)(a-b)\}x+(a-b)(a-c)(b+c)$
$=(a^2+bc)(b+c)$, or $(b^2+c^2+bc-ab-ac)x =$
$(b+c)(a^2+bc)-(a-b)(a-c)(b+c) = (b+c)\{a^2+bc-(a-b)(a-c)\}$
$=(b+c)\{a(b+c)\} = a(b+c)^2$, $\therefore x =$ &c.

42. Reducing as in last example, the coefficients of the third and second powers of x cancel, and we have
$(ab+bc+cd-ad-b^2-c^2)x = bc(d-a)+(a-b)(b-c)(c-d)$, &c.

43. As in last example,

$$x(ab+bc-ac-b^2) = bc^2-b^2c-ac^2+b^2d-abd+acd.$$

44. $x^3-2x^2(a+b+c)+4x(ab+bc+ca)-8abc-x^3+2x^2(a+b+c)-x\{(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)\}+(a+b)(b+c)(c+a) = (a+b+c)(a^2+b^2+c^2)-9abc$, or $-x\{a^2+b^2+c^2+3(ab+bc+ca)+4(ab+bc+ca)x = (a+b+c)(a^2+b^2+c^2)-abc-(a+b)(b+c)(c+a)$. And since $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca)-abc$, the equation becomes
$-x(a^2+b^2+c^2-ab-bc-ca) = (a+b+c)(c^2+b^2+c^2-ab-bc-ca)$,
$\therefore x = -(a+b+c)$.

45. Reduce as in last example, then $x(a^2+b^2+c^2-ab-bc-ca)$ $=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, $\therefore\ x=a+b+c$.

46. $abcx-(ab+bc+ca)+\dfrac{a+b+c}{x}-\dfrac{1}{x^2}+\dfrac{1}{x^2}=\dfrac{a+b+c}{x}$,

$\therefore\ x=(ab+bc+ca)\div abc$.

47. $(2a+b+c)x+a(b+c)=(2d+b+c)x+d(b+c)$,

$\therefore\ 2(a-d)x=(d-a)(b+c),\ x=-\frac{1}{2}(b+c)$.

48. $ax(2b-c-ab)-b(c+ab)=ax(-2ac-ab-c)+ac(ab+ac)$,

$\therefore\ ax(2b+2ac)=(ab+c)(ac+b),\ x=(ab+c)\div 2a$.

49. By division and cancelling,

$$\frac{5}{x-4}+\frac{22}{x-8}=\frac{31x-164}{x^2-12x+32},$$

where the denominators of both members are equal;

$\therefore\ 27x-128=31x-164,\ x=9$.

50. $\dfrac{5}{3}-\dfrac{2}{x+1}-\dfrac{3}{2}-\dfrac{5}{2x-2}=\dfrac{1}{6}-\dfrac{30x-3}{6x^2-6}$

$\therefore\ \dfrac{9x+1}{6(x^2-1)}=\dfrac{10x-1}{6(x^2-1)}$, and $x=2$.

51. Complete the divisions

$$\therefore\ \frac{13}{2x-9}+\frac{6}{x+3}=\frac{22x+6}{2x^2-3x-27},$$

where denominators of both members are equal;

$\therefore\ 25x-15=22x+6,\ x=7$.

52. $\dfrac{45x^2-49x+26}{(3x-2)(3x-1)}=5$; $-4x=-16,\ x=4$.

53. Taking together fractions of like denominators we have

$$\frac{5}{3x-7}-\frac{4}{2x-5}+\frac{3}{9x-25}=0,$$

$\therefore\ 27x^2-10x+30=0$; $x=\frac{1}{27}(5\pm\sqrt{785})$.

54. (*a*) Divide by x, $\therefore x=0$; quotient is

$$\frac{4x-3}{1+x}-\frac{3}{1-x}=\frac{4x^2+2}{x^2-1};\text{ divide}$$

$$-\left(\frac{7}{1+x}+\frac{3}{1-x}\right)=\frac{6}{x^2-1},\quad\therefore x=4.$$

(*b*) $\dfrac{x-a}{x-m}=\dfrac{x-b}{x-n}=\dfrac{b-a}{n-m}$, $x(n-m+a-b)=am-nb.$

If the sign between the quantities is + (as first given in Hand-Book) the equation is a quadratic, and

$$\therefore x=\tfrac{1}{4}\{a+b+m+n\pm\sqrt{\{(a+b+m+n)^2-4(an+bm)\}}.$$

55. (*a*) $x=\frac{1}{12}$. (*b*) Completing divisions,

$$\therefore \frac{a}{c}+\frac{c}{a}+\frac{bc}{a(ax-b)}=\frac{c}{a}+\frac{a}{c}+\frac{ab}{c(cx-b)}$$

$$\therefore \frac{c}{a(ax-b)}=\frac{a}{c(cx-b)}\quad\therefore (a^3-c^3)x=b(a^2-c^2)$$

$$\therefore x=b(a+c)\div(a^2+ab+b^2).$$

56. Multiply terms of first fraction by $2x$, those of second by $3x$, and those of 3rd by $6x$, then

$$\frac{3x-2}{3x+2}-\frac{2x-3}{2x+3}=\frac{9x-4x}{4+6x},\text{ or }\frac{10x}{(3x+2)(2x+3)}=\frac{5x}{4+6x};$$

$\therefore x=0$, and $(3x+2)(2x-1)=0$, or $x=-\frac{2}{3}$, and $\frac{1}{2}$.

57. By division $\dfrac{12}{x-7}+\dfrac{12}{x-4}=\dfrac{132}{5x-28}$; or $\dfrac{2x-11}{(x-7)(x-4)}=$

$$\frac{11}{5x-28}=\frac{2x}{x^2-6x}=\frac{2}{x-6}\;\therefore 11x-66=10x-56,\ x=10.$$

58. (*a*) By division $\dfrac{a}{m}+\dfrac{ap}{m(mx-p)}+\dfrac{c}{n}+\dfrac{cq}{n(nx-q)}=\dfrac{a}{m}+\dfrac{c}{n}$

$$\therefore (apn^2+cqm^2)x=apnq-cmpq,\ \therefore x=\&c.$$

Before dividing, both members may be multiplied by mn to avoid fractional quotients.

$$(b)\quad \frac{a}{m}+\frac{ap+mb}{m(mx-p)}+\frac{c}{n}+\frac{cq+nd}{n(nx-q)}=\frac{a}{m}+\frac{c}{n}$$

$$\therefore x = \frac{nq(ap+mb) - mp(cq+nd)}{apn^2 + mn^2b - m^2cq - m^2nd}.$$

59. (*a*) By division $\frac{a+b}{x+a} + \frac{a-c}{x-c} = \frac{a(2x-c)}{x^2-a^2}$,

$\therefore (a+b+a-c-2a)x = -ac+ab+ac = ab,$

$$\therefore x = ab \div (b-c).$$

(*b*) $\frac{a+b}{x-a} + \frac{b+c}{x-b} = \frac{a+b}{x-c} + \frac{c+b}{x-c}\ \therefore$

$$(a+b)\left(\frac{1}{x-a} - \frac{1}{x-c}\right) + (b+c)\left(\frac{1}{x-b} - \frac{1}{x-c}\right) = 0,$$

or $\frac{(a+b)(a-c)}{(x-a)(x-c)} + \frac{(b+c)(b-c)}{(x-b)(x-c)} = 0$, or $\frac{a^2+(b-c)a-bc}{x-a}$

$+ \frac{b^2-c^2}{x-c} = 0$, or $(a^2+b^2-c^2+ab-bc-ac)x =$

$c\{a^2+(b-c)a-bc\}+a(b^2-c^2)$, $\therefore$ &c.

60. $\frac{ax}{ax-b} + \frac{b}{ax+b} - \frac{bx}{ax+b} = \frac{ax}{ax-b}$ − &c.

$$\therefore \frac{b(ax+b-ax^2+bx)}{a^2x^2-b^2} = -\frac{(ax^2-2b)b}{a^2x^2-b^2}.$$

61. By division, $\frac{a}{m} + \frac{ap-mb}{m(mx-p)} + \frac{c}{n} + \frac{cq-nd}{n(nx-q)} +$

$\frac{(bn+dm)x-(bq+dp)}{(mx-p)(nx-q)} = \frac{a}{m} + \frac{c}{n}$,

$\therefore (apn^2 - cqm^2)x = mpcq + apnq$, $\therefore$ &c.

62. Take together the fractions whose numerators are alike,

thus, $m\left\{\frac{1}{x-a} - \frac{1}{x-c}\right\} + n\left\{\frac{1}{x-b} - \frac{1}{x-a}\right\} +$

$p\left\{\frac{1}{x-c} - \frac{1}{x-b}\right\} = 0$, or $\frac{m(a-c)}{(x-a)(x-c)} + \frac{n(b-a)}{(x-a)(x-b)}$

$+ \frac{p(c-b)}{(x-c)(x-b)} = 0$. Clear and transpose ;

$\therefore\ x\{m(a-c)+n(b-a)+p(c-b)\}$
$= bm(a-c)+cn(b-a)+ap(c-b), \quad \therefore\ x = \&c.$

63. (1) By subtracting numerators and denominators each fraction $= \dfrac{a-b}{a+b}$, $\therefore$ by (5), page 122, $\dfrac{a-b}{ax-2b} = \dfrac{b}{a+b}$,
$\therefore\ x = (a^2+b^2) \div ab.$

(2) Multiply terms of first fraction by ax, and those of second by x, $\therefore\ \dfrac{x-a}{x+a} = \dfrac{ax-1}{ax+1}$, $\quad\therefore\ x=0.$

(3) $\dfrac{2x^2-3x+15}{7x^2-4x+2} = \dfrac{2}{7} = \dfrac{2x^2}{7x^2} = \dfrac{3x-5}{4x-2}$, $\therefore\ x = \dfrac{31}{13}.$

64. (1) Proceed as in last Ex., each fraction $= \dfrac{bx-c}{nx-p}$,
$\therefore\ x = (ap-cm) \div (an-bm).$

(2) As in last Ex. each fraction $=$
$\dfrac{ax-d}{mx-q}$, $\therefore\ x = \dfrac{d(n-q)-q(b-d)}{a(n-q)-m(b-d)}.$

65. (1) Multiply terms of first and third fraction by **four.**
$\dfrac{1-4x}{1+4x} + \frac{1}{4} = \dfrac{4x}{1+4x} - \frac{1}{4}$; $\quad x = \frac{1}{4}.$

(2) Transpose and add, $\therefore\ \frac{4}{3}x \div (\frac{2}{3}-x) = \frac{4}{3}$; $x = \frac{1}{3}.$

66. By transposing, $21\left(\dfrac{1}{x-98} - \dfrac{1}{x+44}\right) =$
$71\left(\dfrac{1}{x-94} - \dfrac{1}{x-52}\right)$, or $\dfrac{21\times142}{(x-98)(x+44)} = \dfrac{71\times42}{(x-94)(x-52)}$,
$\dfrac{1}{\cdots\cdots\cdots\cdots} = \dfrac{1}{\cdots\cdots\cdots\cdots}$ $\therefore$ denominators are equal

$\therefore\ \dfrac{x-94}{x-98} = \dfrac{x+44}{x-52} = \dfrac{138}{46} = \dfrac{3}{1}$, $\quad\therefore\ x=100.$

67. (1). $\dfrac{7}{x-6} - \dfrac{1}{x-12} = \dfrac{9}{x-7} - \dfrac{3}{x-11}$, or $\dfrac{6x-78}{(x-6)(x-12)} =$
$\dfrac{6x-78}{(x-7)(x-11)}$ $\therefore\ 6x-78=0,\ x=13.$

(2). As in 66, take together the fractions where numerators are alike, then $\frac{324}{(x-51)(x-15)} = \frac{324}{(x-81)(x+81)}$.

$$\therefore \frac{x-81}{x-51} = \frac{x-15}{x+81} = \frac{66}{132} = \tfrac{1}{2}, \therefore x = 111.$$

68. (1) $\frac{5}{x-6} - \frac{1}{x-10} = \frac{8}{x-7} - \frac{4}{x-9}$, or $\frac{4x-44}{(x-6)(x-10)} = \frac{4x-44}{(x-7)(x-9)}$ $\therefore$ $4x-44=0$, $x=11$.

(2). $\frac{1}{x-6} - \frac{5}{x-2} = \frac{4}{x-5} - \frac{8}{x-3}$, or $\frac{4x-28}{\dots\dots} = \frac{4x-28}{\dots\dots}$

$\therefore 4x-28=0$, $x=7$.

69. $(m-n)\left(\frac{1}{x-a} - \frac{1}{x-b}\right) = (a-b)\left(\frac{1}{x-m} - \frac{1}{x-n}\right)$,

$\frac{(m-n)(a-b)}{(x-a)(x-b)} = \frac{(a-b)(m-n)}{(x-m)(x-n)}$ $\therefore$ $\frac{x-n}{x-a} = \frac{x-b}{x-m} = \frac{b-n}{m-a}$ (by subtraction), $\therefore x = (a+b-m-n)$.

70. Transpose

$$\frac{a+b}{x-b} + \frac{c+d}{x-(a+b+c+d)-b} = \frac{a+c}{x-c} + \frac{b+d}{x-(a+b+c+d)-c}$$

Combine the two fractions of first member; also those of second,

$$\frac{x(a+b+c+d)-(a+b)(a+b+c+d)-b(a+b)-b(c+d)}{(x-b)\{x-(a+b+c+d)-b\}}$$
$$= \frac{x(a+b+c+d)-(a+c)(a+b+c+d)-c(a+c)-c(b+d)}{(x-c)\{x-(a+b+c+d)-c\}}$$

Divide by $a+b+c+d$

$$\frac{x-a-2b}{(x-b)\{x-(a+b+c+d)-b\}} = \frac{x-a-2c}{(x-c)\{x-(a+b+c+d)-c\}}$$

$= \frac{2(b-c)}{(b-c)\{x-(a+b+c+d)+b(x-b)-c(x-c)}$ by subtraction of numerators $= \frac{(2b-c)}{(b-c)\{x-(a+b+c+d)+x(b-c))-(b+c)b-c)}$,

$$= \frac{2}{x-(a+c+c+d)+x-b-c} \text{ by cancelling } b-c,$$

$$= \frac{2}{2x-a-2b-2c-d}.$$

$$\therefore \frac{x-a-2b}{(x-b)(x-a-2b-c-d)} = \frac{2}{(2x-a-2b-2c-d)},$$

Multiplying by $x-a-2b-c-d$,

$$\frac{2(x-a-2b)}{2(x-b)} = \frac{2x-2a-4b-2c-2d}{2x-a-2b-2c-d} = \frac{2(c-d)}{a+2c+d} \text{ substitution,}$$

$$\therefore \frac{x-a-2b}{x-b} = \frac{2c+2d}{a+2c+d} \text{ invert terms}$$

$$\frac{x-b}{x-a-2b} = \frac{a+2c+d}{2c+2d} \quad \therefore \frac{x-b}{a+b} = \frac{a+2c+d}{a-d},$$

by subtracting terms $x-b = \dfrac{(a+b)(a+2c+d)}{a-d}$

$$\therefore x = \frac{a^2+2ac+ad+ab+2bc+bd+ab-bd}{a-d}$$

$$= \frac{a^2+2ac+ad+2bc+2ab}{a-d}.$$

71. The first number vanishes if $x-a=0$, and since it is symmetrical with respect to x, a, b, it will also vanish if $a-b=0$, or $b-x=0$. Since the expression is of three dimensions, $x-a$, $x-b$ and $b-x$ are the literal factors of it. Put $x=0$, $a=1$, $b=2$, and we get the numerical factor -6. $\therefore$ the $=n$ becomes

$$6(x-a)(a-b)(b-x)=(a-b)c^2.$$

$$(x-a)(b-x) = \frac{c^2}{6}, \quad x^2-x(a+b)+ab = -\frac{c^2}{6},$$

$$x^2-x(a+b)+\frac{(a+b)^2}{4} - \left\{\frac{(a-b)^2}{4} - \frac{c^2}{6}\right\} = 0,$$

$$\left\{x - \frac{a+b}{2} + \sqrt{\frac{(a-b)^2}{4} - \frac{c^2}{6}}\right\}\left\{x - \frac{a+b}{2} - \sqrt{\frac{(a-b)^2}{4} - \frac{c^2}{6}}\right\} x$$

$$= \frac{a+b}{2} \pm \sqrt{\frac{(a-b)^2}{4} - \frac{c^2}{6}}$$

72. Both members vanish if $x=0$, $\therefore$ this is one solution. Since first member is symmetrical with respect to the letters x, a, b, $\therefore$ it will also vanish if $a=0$ or $b=0$, $\therefore$ abx is a factor. One factor of two dimensions remains, which must similarly involve x, a, b. It must therefore have the form $m(x^2+a^2+b^2)$ $+n(xa+ab+bx)$. We have to find m and n. Writing first part thus, $\{x+(a+b)\}^5$, and expanding, we get

$$x^5+5x^4(a+b)+10x^3(a+b)^2+10x^2(a+b)^3, \text{ \&c.}$$

$10x^3(a+b)^2$ has for its second term $20abx^3$ or $20x^2(abx)$. This is the only term of the expansion of the first member containing abx^3, $\therefore m=20$. Also second term of $10x^2(a+b)^3$ is $30x^2a^2b$ or $30xa(abx)$, $\therefore n=30$. Therefore the equation becomes

$$10abx(2x^2+2a^2+2b^2+3xa+3ab+3bx)=$$

$$10abx(2x+a+b)(x+a+b). \qquad \text{Cancel } 10abx,$$

$$2x^2+2a^2+2b^2+3x(a+b)+3ab=2x^2+3x(a+b)+(a+b)^2,$$

which gives no result, since the terms involving x vanish,

$\therefore$ $x=0$ is the only solution.

73. $$\frac{(m-n)(x-a)}{b+c}+\frac{(n-p)(x-b)}{c+a}+\frac{(p-m)(x-c)}{a+b}=0.$$

Also $(m-n)+(n-p)+(p-m)=0$, identically. Subtract this, term by term, from the left-hand member,

$$\therefore (m-n)\left(\frac{x-a}{b+c}-1\right)+\text{\&c.}=0,$$

$\therefore$ $x-(a+b+c)$ is a factor, and $x=a+b+c$.

74. Transpose and arrange thus:

$$x\left\{\left(\frac{a}{a^2(b+c)}-\frac{1}{ab+bc+ca}\right)+\left(\frac{b}{b^2(c+a)}-\frac{1}{ab+bc+ca}\right)+\right.$$

$$\left.\left(\frac{c}{c^2(a+b)}-\frac{1}{ab+bc+ca}\right)\right\}=\frac{1}{a^2(b+c)}+\frac{1}{b^2(c+a)}+\frac{1}{c^2(a+b)}$$

$$\frac{abcx}{ab+bc+ca}\left\{\frac{1}{a^2(b+c)}+\frac{1}{b^2(c+a)}+\frac{1}{c^2(a+b)}\right\}=$$

$$\frac{1}{a^2(b+c)}+\frac{1}{b^2(c+a)}+\frac{1}{c^2(a+b)} \quad \therefore \quad \frac{abcx}{ab+bc+ca}=1$$

$$x = \frac{ab+bc+ca}{abc}.$$

75. Transpose and arrange thus:

$$\left(\frac{x-2a}{b+c-a}-1\right)+\left(\frac{x-2b}{c+a-b}-1\right)+\left(\frac{x-2c}{a+b-c}-1\right)=0$$

$$\frac{x-a-b-c}{b+c-a}+\frac{x-a-b-c}{c+a-b}+\frac{x-a-b-c}{a+b-c}=0$$

$\therefore\ x-a-b-c=0,\ x=a+b+c.$

76. Transpose and arrange as in last example,

$$\frac{x-2a}{b+c-a}-1+\&c. = 3\left(\frac{x}{a+b+c}-1\right).$$

77. $\dfrac{a-x}{a^2-bc}-\dfrac{1}{a+b+c}+\&c.+\&c=0$, or

$$\frac{ab+bc+ca-x(a+b+c)}{(a^2-bc)(a+b+c)}+\text{anal}+\text{anal}=0,\ \therefore$$

$x=(ab+bc+ca)\div(a+b+c).$

78. Combine the fractions in pairs as they stand.

$$\frac{-2ax+4ab(b+c)}{(b+c+a)(b+c-a)}=\frac{2ax-4ab(b-c)}{(a-\{b-c\})(a+\{b-c\})}$$

$$\frac{-x+2b(b+c)}{(b+c)^2-a^2}=\frac{x-2b(b-c)}{a^2-(b-c)^2}=\frac{4bc}{4bc}=1 \text{ by addition,}$$

$\therefore\ x=2b^2-2bc+a^2-b^2+2bc-c^2=b^2+a^2-c^2.$

79. $\dfrac{a}{x+b-c}+1+\dfrac{b}{x+a-c}+1=\dfrac{a-c}{x+b}+1+\dfrac{b-c}{x+a}+1$, or

$$\frac{x+a+b-c}{x+b-c}+\frac{x+a+b-c}{x+a-c}=\frac{x+a+b-c}{x+b}+\frac{x+a+b-c}{x+a},$$

$\therefore\ x=c-a-b.$

81. Let $x^2-11x=z$, $a^2-9a=n$, and $b^2+13=m$, then the equation reduces to

$$(z+18)(z+30)=-(m+12)(m+40)-(n-22)(n+20)+$$
$$(m+22)(m+30)+(n-10)(n+8), \text{ or } 48z+540=540,$$
$$\therefore z=0, \; i.e., \; x^2-11x=0, \; \therefore \; x=0; \text{ or } 11.$$

Exercise lvi., page 172.

1. $A=0$, or $B=0$. 2. $A=0$, or $B=0$, or $C=0$.

3. $x=0$, or $a-b=0$. 4. $x=0$, or $y=0$.

5. In the first case either $x-5y=0$, *or* $x-4y+3=0$; in the second case both conditions hold.

6. $x=0$, or $x=a$. 7. $x=0$, or $x=-b$.

8. $x=a$, or $x=c\div b$. 9. $x=0$, or $x=3$.

10. $x=0$, or $x=a+b$. 11. $x=0$, or $x^2=a^2$, $\therefore \; x=\pm a$.

12. $x=0$, or $x^2=\dfrac{b^2}{a^2}\left(\therefore \; x=\dfrac{b}{a}\right)$.

13. $x=0$, or $x=a$. 14. $x=0$, or $x=a$.

15. $x=0$, or $x=a+b$. 16. $x=0$, or $a+b$.

17. $-(a+b)x+ab=ab$, $x=-(2ab)\div(a+b)$.

18. $(x-a)(x-b)=0$, $\therefore \; x=a$, or b.

19. $(x-a)(x-b)(x-c)=0$, $x=a$, or b, or c.

20. $x=5$. 21. $(x+30)(x-1)=0$, $x=1$.

22. $(x-21)(x+4)=0$, $\therefore \; x=21$.

23. $(3x-1)(x-3)=0$, $x=\frac{1}{3}$, $x=3$.

24. $(x-9)(x-4)=0$, $x=9$, $x=4$.

25. $(x-1)(x+2)(x-3)=0$, $\therefore \; x=1$, or 3.

26. $-2(a+b)x=-2ab$, $\therefore \; x=(ab)\div(a+b)$.

27. $3x^2-3(a+b)x+3ab=0$, or $x^2-(a+b)x+ab=0$,
$\therefore\ (x-a)(x-b)=0$, and $x=a$, or b.

28. $a^2(a-x)^2-b^2(b-x)^2=0$, then $a(a-x)+b(b-x)=0$,
or $a(a-x)+b(b-x)=0$, $x=(a^2+b^2)\div(a+b)$, $x=b+a$

29. $a^2(b-x)^2-b^2(a-x)^2=0$, $\therefore\ a(b-x)+b(a-x)=0$,
or $a(b-x)-b(a-x)=0$, $x=(2ab)\div(a+b)$.

30. $(x-a)^3+(a-b)^3+(b-x)^3=3(x-a)(a-b)(b-x)=0$,
$\therefore\ x=a$, or b.

31. $(x-1)^2-a(x^2-1)=0$, or $(x-1)(x+1-ax-a)=0$,
$\therefore\ x=1$, or $(1+a)\div(1-a)$.

32. $(x-a)(x-b)-(x-a)(c+x)=0$, $\therefore\ x=a$,

34. $(x-a+b)(x-a+c)=-(x-a+b)(x+a-b)$,
$\therefore\ x=a-b$, or $\frac{1}{2}(b+c)$.

35. $(x-a-b)\{(x-a+b)-(b+c-x)\}=0$,
$(x-a-b)(2x-a-c)=0$, $x=a+b$, or $\frac{1}{2}(a+c)$.

36. $\{(a+b+c)x-a\}\{x-1\}=0$, $\therefore\ x=\dfrac{a}{a+b+c}$ or 1.

37. $\dfrac{a+b-x}{c}=\dfrac{a+b-c}{x}=\dfrac{x-c}{x-c}=1$, $\therefore x=a+b-c$.

38. $(a-x)^2+(a-b)^2=\{(a-x)+(b-x)\}^2$,
$\therefore\ (a-b)^2-(b-x)^2-2(a-x)(b-x)=0$,
or $(a-x)(a-2b+x)-2(a-x)(b-x)=0$,
$(a-x)(a-4b+3x)=0$. $\therefore\ x=a$, or $\frac{1}{3}(4b-a)$.

39. $(a+b)(x+c)-(x^2-c^2)=0$
$\therefore\ (x+c)(a+b+c-x)=0$, $\therefore\ x=-c$, or $a+b+c$.

40. $x(x^2-1)-px(x-1)-m(x-1)=0$
$(x-1)(nx+n-px-m)=0$, $\therefore\ x=1$, or $\dfrac{m-n}{n-p}$.

41. $\dfrac{ax^2-bx+c}{mx^2-nx+p} = \dfrac{c}{p} = \dfrac{ax^2-bx}{mx^3-nx}$

$\dfrac{ax-b}{mx-n} = \dfrac{c}{p} \therefore x = \dfrac{nc-pb}{mc-ap}$

42. $\dfrac{ax^2-bx+c}{mx^2-nx+p} = \dfrac{a-b+c}{m-n+p} = \dfrac{a(x^2-1)-b(x-1)}{m(x^2-1)-n(x-1)}$

$\therefore \dfrac{ax+a-b}{mx+m-n} = \dfrac{a-b+c}{m-n+p} = \dfrac{ax-c}{mx-p}$

$$x=\frac{p(a-b)-c(m-n)}{m(c-b)-a(n-p)}.$$

43. By transposing, $4x^2+a^2-b^2-2(a+b)x-2(a-b)x=0$.
$4x^2+a^2-4ax-b^2=0$, or $(2x-a)^2-b^2=0$,
$\therefore (2x-a-b)(2x-a+b)=0$, $x=\frac{1}{2}(a+b)$, or $\frac{1}{2}(b-a)$.

44. $(2a-b-x)^2=(a+b-2x)^2-9(a-b)^2$
$=(4a-2b-2x)(4b-2a-2x)=4(2a-b-x)(2b-a-x)$,
$\therefore (2a-b-x)(3x+ba-9b)=0$, and $x=2a-b$, or $3b-2a$.

45. $\dfrac{2a+2c-x}{2b+x} = \dfrac{3a-b+3c-2x}{2a+2c-x}$

$\dfrac{2(a+c-b-x)}{2b+x} = \dfrac{(a+c-b-x)}{2a+2c-x)}$,

$\therefore (a+c-b-x)(4a+4c-3x-2b)=0$, and

$x=a+c-b$, or $x=\dfrac{4a+4c-2b}{3}$.

46. $\dfrac{7a-b-3x}{3a-5b+x} = \dfrac{5a-3b-x}{7a-b-3x}$, $\therefore$ by subtraction of terms,

$\dfrac{4(a+b-x)}{3a-5b+x} = \dfrac{-2(a+b-x)}{7a-b-3x}$, $(a+b-x)(17a-7b-5x)=0$,

$$x=a+b, \text{ or } \frac{17a-7b}{5}.$$

47. $\dfrac{3a-b+x}{5a+3b-3x} = \dfrac{5a+3b-3x}{3a+b-x}$, $\therefore$ as in last Ex.,

$\dfrac{8a+2b-2x}{5a+3b-3x} = \dfrac{8a+2b-2x}{3a+b-x}$, $\therefore x=4a+b$, or $a+b$.

48. $c(b-c)x^2-b(a-b+b-c)x+a(a-b)=0$,

$\{(b-c)x-(a-b)\}\{cx-a\}=0$,

$x = \dfrac{a-b}{b-c}$, or $\dfrac{a}{c}$.

49. Equation reduces to

$(b-c)(a-b)x^2-(a^2+b^2+c^2-ab-bc-ca)x+(a-c)(a-b)=0.$

50. Equation reduces to (use formula B)

$2(x^4-13x^2+84)=96$, or $x^4-13x^2+36=0$,

or $(x^2-9)(x^2-4)=0$, $\therefore x=\pm3$, or ±2.

51. Proceeding as in the last Ex., the equation reduces to

$2(x^4-40x^2+135)+18=0$, or $x^4-40x^2+144=0$,

$\therefore (x^2-36)(x^2-4)=0$, and $x=\pm6$, or ±2.

52. $x-3+\dfrac{1}{x}-\dfrac{1}{3}=0$, $\therefore (x-3)\left(1-\dfrac{1}{3x}\right)=0$,

$\therefore x=3$, or $\dfrac{1}{3}$.

53. $x-\dfrac{a+b}{a-b}+\dfrac{1}{x}-\dfrac{\frac{1}{a+b}}{a-b}=0$, $\therefore$

$\left(x-\dfrac{a+b}{a-b}\right)\left\{1-\dfrac{1}{x\left(\frac{a+b}{a-b}\right)}\right\}=0$,

$x=\dfrac{a+b}{a-b}$ or $x=\dfrac{a-b}{a+b}$.

54. $x-\dfrac{a}{b}-\left(\dfrac{1}{x}-\dfrac{1}{\frac{a}{b}}\right)=0$, $x=\dfrac{a}{b}$, or $-\dfrac{b}{a}$.

55. $\dfrac{a+x}{b+x}-2+\dfrac{1}{\dfrac{a+x}{b+x}}-\dfrac{1}{2}=0,$

$\therefore \left\{\dfrac{a+x}{b+x}-2\right\}\left\{1-\dfrac{1}{2\left(\dfrac{a+x}{b+x}\right)}\right\}=0,$ and

$x=b-2a$, or $a-2b$.

56. Let $z=\dfrac{a-x}{x-b}$, the equation becomes $z-\dfrac{3}{2}+\dfrac{1}{z}-$ $\dfrac{2}{3}=0$, or $(z-\frac{3}{2})\left(1-\dfrac{1}{\frac{3}{2}z}\right)=0$, $\therefore z=\frac{3}{2}$, and $\frac{3}{2}z=1$.

Restore values of z, and $x=\dfrac{2a+3b}{5}$, or $\dfrac{3a+2b}{5}$.

57. $\dfrac{a-x}{b+x}-\dfrac{m}{n}-\dfrac{1}{\dfrac{a-x}{b+x}}+\dfrac{1}{\dfrac{m}{n}}=0$, $\therefore$ as in former cases,

$x=(mb+na)\div(m+n)$, or $(ma-nb)\div(m+n)$.

58. $\dfrac{a}{x}+\dfrac{x}{a}=\dfrac{m}{n}$, $\quad\therefore \dfrac{a^2+x^2}{2ax}=\dfrac{m}{2n}$,

$\therefore \dfrac{(a+x)^2}{(a-x)^2}=\dfrac{m+2n}{m-2n}$, or $\dfrac{a+x}{a-x}=\sqrt{\left(\dfrac{m+2n}{m-2n}\right)}$,

$\therefore x=\sqrt{\{(m+2n)}-\sqrt{(m-2n)\}}\div\sqrt{\{(m+2n)}+\sqrt{(m-2n)\}}$.

59. $\dfrac{x^2+ax+a^2}{x^2-ax+a^2}=c \quad\therefore \dfrac{x^2+a^2}{2ax}=\dfrac{c}{1}$,

$\therefore \dfrac{x+a}{x-a}=\dfrac{\sqrt{(c+1)}}{\sqrt{(c-1)}}$, and $x=a\left\{\dfrac{\sqrt{(c+1)}+\sqrt{(c-1)}}{\sqrt{(b+1)}-\sqrt{(c-1)}}\right\}$,

60. $\dfrac{x^2+a^2}{2ax}=\dfrac{c}{2(c-1)} \quad\therefore \dfrac{x+a}{x-a}=\dfrac{\sqrt{(3c-2)}}{\sqrt{(2-c)}}$

$\therefore x=a\{\sqrt{(3c-2)}+\sqrt{(2-c)}\}\div\{\sqrt{(3c-2)}-\sqrt{(2-c)}\}$.

61. $\dfrac{(x-a)^2}{(x+a)^2}=\dfrac{2c-1}{1}$, $\dfrac{x-a}{x+a}=\dfrac{\sqrt{(2c-1)}}{1}$

$\therefore x=a\{\sqrt{(2c-1)}+1\}\div\{1-\sqrt{(2c-1)}\}$.

62. $\dfrac{\{(a-x)+(x-b)\}^2}{\{(a-x)-(x-b)\}^2} = \dfrac{9}{1} \therefore \dfrac{a-b}{a+b-2x} = \dfrac{3}{1}$

and $x = \frac{1}{3}(a+2b)$.

63. $\dfrac{(a-x)^2+(x-b)^2}{(x-b)(a-x)} = \dfrac{m}{n}$, which is exactly the form in last example, $\dfrac{a-b}{a+b-2x} = \dfrac{\sqrt{(m+2n)}}{\sqrt{(m-2n)}}$

$$2x = (a+b)\sqrt{(m+2n)} - (a-b)(m-2n)$$

64. $\left(\dfrac{a+b}{x+b}\right)^2 = \left(\dfrac{a+b}{a-b}\right)^2 \therefore \dfrac{x+a}{x+b} = \pm\dfrac{a+b}{a-b}$

$$\therefore x = -\frac{a^2+b^2}{2b}, \text{ or } -\frac{a^2+b^2}{2a}.$$

65. As the equation stands it reduces to a quadratic; but change sign between the squares in numerator of left-hand member, and between the squares in denominator of right-hand member, and it can be solved as 64.

66. $\dfrac{(a-x)^2+(x-b)^2}{x(a-x)(x-b)} = \dfrac{34}{15}, \therefore \dfrac{(a-x+x-b)^2}{(a-x-x+b)^2} = \dfrac{64}{4}$,

or, $\dfrac{a-b}{a+b-2x} = \dfrac{8}{2} = \dfrac{4}{1}$, or $-\dfrac{4}{1}$,

$x = \dfrac{3a+5b}{8}$, or $\dfrac{3b-5a}{8}$.

67. $\dfrac{2a^2+a(a-x)+(a+x)^2}{2a^2+a(a+x)+(a-x)^2} = \dfrac{c+1}{c-1}$,

or, $\dfrac{4a^2+ax+x^2}{4a^2-ax+x^2} = \dfrac{c+1}{c-1}$, $\quad\therefore \dfrac{4a^2+x^2}{ax} = \dfrac{c}{1}$,

$\therefore \dfrac{(2a+x)^2}{(2a-x)^2} = \dfrac{c+4}{c-4}$, or $\dfrac{2a+x}{2a-x} = \dfrac{\sqrt{(c+4)}}{\sqrt{(c-4)}}$

$$x = 2a\,.\,\{\sqrt{(c+4)} - \sqrt{(c-4)}\} \div \{\sqrt{(c+4)} + \sqrt{(c-4)}\}.$$

68. Let $u=5-x$, $v=x-2$ (See example 12, page 170),

$$u+v=3,\quad u^4+v^4=17$$

$$(u+v)^4=u^4+v^4+4(u+v)^2uv-2u^2v^2,$$

$$81=17+36(5-x)(x-2)-2\{(5-x)(x-2)\}^2,$$

$$\{(5-x)(x-2)\}^2-18(5-x)(x-2)+81=49$$

$$(5-x)(x-2)=16 \text{ or } 2, \therefore x=4, \text{ or } 3.$$

69. (1). $u=x$, $v=a-x$,

$u+v=a$, $u^4+v^4=c$. Proceed as in last example and we find $x^2(a-x)^2-2a^2(a-x)x+a^4=$

$$\frac{c+a^4}{2} \therefore x(a-x)=a^2\pm\sqrt{\frac{c+a^4}{2}}=m \text{ suppose, \&c.}$$

(2). Substitute in the values of x found in the last example, $a=4$, and $c=82$, we find $x=3$, or 1.

70. Substituting as in last examples, we find

$$(a-b)^4=(a-b)^4+4(a-b)^2(a-x)(x-b)-2(a-x)^2(x-b)^2$$

or $(a-x)(x-b)\{(a-x)(x-b)-2(a-b)^2\}=0$,

$\therefore x_1=a$ and $x_2=b$; and also $x^2-(a+b)x+(2a^2-ab+2b^2)=0$.

71. Let $u=a-x$, $u+v=a-b$;

$v=x-b$, $u^5+v^5=c$;

$\therefore (u+v)^5=u^5+v^5+5uv(u+v)^3-5u^2v^2(u+v)$,

or, $(a-b)^5=c+5(a-b)^3(a-x)(x-b)-5(a-b)(a-x)^2(x-b)^2$.

Let $(a-x)(x-b)=t$, and $\dfrac{c-(a-b)^5}{5(a-b)}=r$,

Then $t^2-(a-b)^2t-r=0$, $t=\dfrac{(a-b)^2\pm\sqrt{\{(a-b)^4+4r\}}}{2}$;

But $(x-a)(x-b)=-t$, *i.e.*, $x^2-(a+b)x+ab+t=0$,

$$x=\frac{a+b\pm\sqrt{(a+b)^2-4(ab+t)}}{2}.$$

72. (1) Let $u=x$, $u+v=a$, $v=a-x$, $u^5+v^5=a^5$, and proceed as in last example, then

$$a^5=a^5+5a^3(a-x)x-5a(a-x)^2x^2,$$

$$(a-x)x\{x(a-x)-a^2\}=0, \therefore x=0, \text{ or } a, \text{ or } \tfrac{1}{2}a(1\pm\sqrt{-3}).$$

(2) Proceeding exactly as in last example, we find

$$7776=1056+1080(6-x)x-30x^2(6-x)^2$$

$$x^2(6-x)^2-36x(6-x)=-224, \therefore x=4, \text{ or } 2.$$

73. Equation is $(a-x)^2(x-b)^2(a-x+x-b)=a^2b^2(a-b)$

$$\therefore (a-x)(x-b=\pm ab, \therefore x=0,$$

$$\text{or } a+b, \text{ or } \tfrac{1}{2}\{(a+b)\pm\tfrac{1}{2}\sqrt{(a-b)^2-4ab}\}.$$

74. Let $a-x=m$, $b+x=n$, and $\therefore m+n=a+b$; the equation reduces to $mn\{m^3+n^3+mn(m+n)\}=(m+n)c$, or

$$mn\{(m+n)^3-2mn(m+n)\}=(m+n)c,$$

$$\therefore mn\{(m+n)^2-2mn\}=c, \text{ or } (m+n)^2mn-2m^2n^2=c,$$

$$i.\ e., (a-x)^2(b+x)^2-\tfrac{1}{2}(a+b)^2(a-x)(b+x)=-\frac{c}{2},$$

$$\therefore (a-x)(b+x)=\tfrac{1}{4}(a+b)^2\pm\tfrac{1}{4}\sqrt{\{(c+b)^4-8c\}}=r$$

suppose $\therefore x^2-(a-b)x+ab=r$, &c.

75. Let $a-x=m-z$, $x-b=m+z$, and $\therefore a-b=2m$; the equation becomes $\dfrac{m^4+6m^2z^2+z^4}{m^2+z^2}=\dfrac{41}{20}\times 4m^2$ or $5z^4-11m^2z^2=36m^4$,

$$\therefore (5z^2-9m^2)(z^2-4m^2)=0, \therefore z^2=m^2, \text{ and } z=\pm m \text{ and}$$

$$z=\pm 2m, \therefore a-x=+m, \text{ or } 3m, =-\tfrac{1}{2}(a-b), \text{ or}$$

$$\tfrac{3}{2}(a-b), \therefore x=\tfrac{1}{2}(3a-b), \text{ or } \tfrac{1}{2}(3b-a).$$

76. As in last example let $a-x=m-z$, $x-b=m+z$, and $\therefore$ $a-b=2m$; then substitueing these values the equation becomes $\dfrac{m^5+10m^3z^2+5mz^4}{m^4+6m^2z^2+z^4}=\dfrac{211}{97}\times 2m$, or clearing and transposing

$$63z^4-1562m^2z^2-325m^4=0, \ i.e.,$$

$$(63z^2+13m^2)(z^2-25m^2)=0, \therefore z^2=25m^2, \text{ and } z=\pm 5m.$$

And $a-x=m-z=-4m$, or $6m$, $\therefore x=3a-2b$, or $3b-2a$.

77. Make same substitutions as in last two examples, and a quadratic in z^2 is found, $\therefore z$ is known $= t$ suppose,

$$\therefore a-x=m-z=m-t=\tfrac{1}{2}(a-b)-t, \text{ \&c.}$$

78. From same substitutions as last, z is found $=t$, suppose, &c.

79. As in example 76, a quadratic in z^2 is found, $\therefore z=t$, suppose, &c.

80. $\{(a-x)^4+(x-b)^4\} \div (a-x)(b-x)=(a^4+b^4)\div ab.$

Make same substitutions as in example 75, and left-hand member becomes $(z^4+6m^2z^2+m^4)\div(z^2-m^2)$, a quadratic in z^2, then proceed as before.

81. By same substitutions, left-hand member becomes

$$2(m^3+3mz^2)\div(m^2-z^2)^2,$$

which gives a quadratic in z^2.

82. As in former examples, we have left-hand member

$$(z^4+6m^2z^2+m^4)\div 4z^2,$$

and right-hand member is known; a quadratic in z^2.

83. As in example 76,

$$(5mz^4+10m^3z^2+m^5)\div 4z^2=\text{\&c.},$$

a quadratic in z^2.

Note.—For full solutions of these and following equations see Appendix.

CHAPTER VI.

Exercise lvii., page 182.

1. $x=7,\ y=9.$ 2. $x=2,\ y=1.$ 3. $x=8,\ y=1.$

4. $x=9,\ y=5.$ 5. $x=-10\frac{1}{2},\ y=5\frac{1}{2}.$

6. $x=-2,\ y=\frac{1}{2}.$ 7. $x=-1,\ y=1.$ 8. $x=-2,\ y=-3.$

9. $x=-\frac{3}{4},\ y=\frac{1}{2}.$ 10. $x=-\frac{1}{4},\ y=\frac{2}{3}.$

11.
$$\begin{array}{cccc}
\frac{1}{3} & \frac{1}{4} & -6 & \frac{1}{3} \\
3 & -4 & -4 & 3 \\
\hline
-\frac{4}{3} & -1 & -18 & \\
\frac{3}{4} & +24 & -\frac{4}{3} & \\
\hline
-\frac{25}{12} & -25 & -\frac{200}{12} &
\end{array}
\qquad x=12, \quad y=8.$$

12.
$$\begin{array}{cccc}
\frac{1}{2} & -\frac{1}{3} & -1 & \frac{1}{2} \\
\frac{1}{8} & -\frac{2}{3} & 5 & \frac{1}{8} \\
\hline
-\frac{1}{3} & -\frac{5}{3} & -\frac{1}{8} & \\
-\frac{1}{24} & +\frac{2}{3} & 2\frac{1}{2} & \\
\hline
-\frac{7}{24} & \big| -\frac{7}{3} & -\frac{21}{8} & \\
\hline
 & 8 & -9 & \\
 & \parallel & \parallel & \\
 & x & y &
\end{array}$$

13. $x=10,\ y=12.$ 14. $x=12,\ y=15.$

15. $x=18,\ y=13.$ 16. $x=\cdot 3,\ y=\cdot 2.$

17.
$$\begin{array}{cccc}
5 & -4 & 1 & 5 \\
1\cdot 7 & -2\cdot 2 & 7\cdot 9 & 1\cdot 7 \\
\hline
-11 & -31\cdot 6 & & -1\cdot 7 \\
-6\cdot 8 & -2\cdot 2 & & 39\cdot 5 \\
\hline
-4\cdot 2 &)\ -29\cdot 4 & & -37\cdot 8 \\
\hline
 & 7 & & 9 \\
 & \parallel & & \parallel \\
 & x & & y
\end{array}$$

18. $x=7,\ y=\cdot 3.$ 19. $x=7,\ y=3.$

20.

$$\begin{array}{cccc}
1 & 1 & -\frac{5}{6} & 1 \\
1 & -1 & -\frac{1}{6} & 1 \\
\hline
 & -1 & -\frac{1}{6} & \frac{5}{6} \\
 & 1 & +\frac{5}{6} & -\frac{1}{6} \\
 & -2 & \multicolumn{1}{|c}{-1} & -\frac{2}{3} \\
 & & +\frac{1}{2} & \frac{1}{3} \\
 & & \| & \| \\
 & & \frac{1}{x} & \frac{1}{y}
\end{array}$$

$$x=2,\quad y=3.$$

21.

$$\begin{array}{cccc}
3 & 8 & -3 & 3 \\
15 & -4 & -4 & 15 \\
\hline
 & -12 & -32 & -45 \\
 & 120 & 12 & -12 \\
 & -132 & \multicolumn{1}{|c}{-44} & -33 \\
 & & \frac{1}{3} & \frac{1}{4} \\
 & & \| & \| \\
 & & \frac{1}{x} & \frac{1}{y}
\end{array}$$

$$x=3 \qquad y=4.$$

22. $x=\frac{4}{5}\ \ y=\frac{9}{10}.$ 23. $x=\cdot 3\ \ y=\frac{1}{7}.$ 24. $x=12\ \ y=15.$

25.

$$\begin{array}{cccc}
\frac{50}{7} & \frac{3}{10} & -6 & \frac{50}{7} \\
\frac{10}{7} & 9 & -31 & \frac{10}{7} \\
\hline
\frac{450}{7} & -\frac{93}{10} & & -\frac{60}{7} \\
\frac{3}{7} & -54 & & -\frac{1550}{7} \\
\hline
\frac{447}{7} & \multicolumn{1}{|c}{\frac{447}{10}} & & \frac{1490}{7} \\
 & \frac{7}{10} & & \frac{10}{3} \\
 & \| & & \| \\
 & x & & \frac{1}{y}
\end{array}$$

$$x=\frac{7}{10} \qquad y=\frac{3}{10}$$

26. $x=8, \quad y=9$. 27. $x=3, \quad y=1$. 28. $x=7, \quad y=8$.

29. $x=11, \quad y=7$. 30. $x=17, \quad y=13$. 31. $x=5, \quad y=-4$.

32. $x=-\frac{31}{10}, \quad y=-\frac{19}{15}$. 33. $x=13, \quad y=10$.

34. $x=4\frac{2}{5}, \quad y=3\frac{3}{10}$. 35. $x=11, \quad y=6$. 36. $x=7, \quad y=5$.

37.
$$\begin{array}{rrrr}
20 & -15 & 5 & 20 \\
8 & -12 & 20 & 8 \\
\hline
-240 & -300 & & 40 \\
-120 & -60 & & 400 \\
\hline
-120 & \multicolumn{1}{|r}{-240} & & -360 \\
\cline{2-4}
 & 2 & & 3 \\
 & = & & = \\
 & x & & y
\end{array}$$

38. $x=5, \quad y=3$.

39. In this question the second equation is not a new one, but an equation that can be found from the other, hence x and y can not be found.

40. $$1+\frac{2}{x+1}=1+\frac{3}{y+5},$$
$$2\tfrac{1}{2}+\frac{2}{y+1}=2\tfrac{1}{2}+\frac{3}{2x-3}$$
$3x-2y-7=0, \quad 4x-3y-9=0, \quad x=3, \quad y=1$.

41. $$1-\frac{1}{x-3}=1-\frac{3}{y+7}, \quad 1+\frac{3}{x+2}=1+\frac{1}{y-2},$$
$3x-y-16=0, \quad x-3y+8=0, \quad x=7, \quad y=5$.

42. $x=0, \quad y=0$, 43. $4x=7y, \quad x=0, \quad y=0$,

44. $x=0. \quad y=0, \quad x=13, \quad y=\frac{26}{11}$.

45. Adding three equations, $x+y+z=42$.
Subtracting each, $x=17, \quad y=20, \quad z=5$.

46. $x=\frac{221}{130}, \quad y=\frac{234}{130}, \quad z=\frac{247}{130}$. 47. $x=11, \quad y=7, \quad z=9$.

48. Multiplying second $=n$. by 2 and adding
$15x=315, \quad x=21, \quad y=22, \quad z=23$.

49. From first equation $2x=5y$; adding second and third equation, $\frac{1}{3}x-\frac{1}{3}y=-3$, $x=-15$, $y=-6$, $z=-8$.

50. Multiplying first and second, and subtracting second equation from the result, $3y-2\frac{2}{3}z=0$; $y=\frac{4}{5}z$.

Substituting in third equation, z, 5 ; y, 4 ; x, 3.

51. Adding three equations $x+y+z=37$.

Subtracting each equation, x, 12 ; y, 15 ; z, 10.

52. Adding three equations and afterward multiplying second equation by 3.

$$\begin{array}{l} 3x+6y+14z=47 \\ \underline{3x+6y+12z=45} \end{array}$$

$$z=1,\ x=5,\ y=3.$$

53. x, $\frac{5}{6}$; y, $1\frac{1}{2}$; z, $\frac{2}{3}$.

54. Multiplying second equation by 7 and subtracting from first equation, $20y=100$; y, 5 ; x, 3 ; z, 7.

55. Adding all three equations, $10x=110$; x, 11 , y, 13 ; z, 17.

56. Adding three equations, and then multiplying second equation by 3,

$$\begin{array}{l} 3x+6y+10z=43 \\ \underline{3x+6y+\ \ 9z=42} \\ z=1,\ y=3,\ x=5. \end{array}$$

57. x, 9 ; y, 7 ; z, 3.

58. Adding three equations, and then multiplying third equation by 4,

$$\begin{array}{l} 4x+4y+4z=99 \\ \underline{8x+4y+4z=128} \\ 4x=29,\ x=7\frac{1}{4},\ y=8\frac{1}{4},\ z=9\frac{1}{4}. \end{array}$$

59. Adding three equations,

$$\begin{array}{rr} & 21(x+y+z)=135 \\ \text{From (3)} & \underline{7x+21y+21z=\ \ 91} \end{array}$$

$$14x=44 ;\ x,\ 3\frac{1}{7};\ y,\ 2\frac{1}{7};\ z,\ 1\frac{1}{7}.$$

60. Adding 1 and 3, multiplied by 2, and again adding second and third, we have two equations in x and z.

$$x,\ 2\cdot3\ ;\ y,\ 3\cdot4\ ;\ z,\ 4\cdot5.$$

61. x, 30 ; y, 20 ; z. 70.

62. x, $\frac{880}{59}$; y, $\frac{1098}{59}$; z, $\frac{1004}{59}$;

63. x, 30 ; y, 12 ; z, 70.

64. x, 6 ; y, 12 ; z, 20.

65. x, 5 ; y, 2 ; z, 0.

66. x, 1 ; y, 1 ; z, 1

67. x, 11 ; y, 9 ; z, 7.

68. x, 5 ; y, 3 ; z, 1.

69. Multiplying the first and third equations by 3, and adding to these results the second equation,

$$\frac{14}{x} = 7,\ x=2,\ y=3,\ z=1.$$

70. Changing sign of third equation, and adding all three,

$$\frac{20}{z} = 4.\quad z=5,\quad y=4,\quad x=3.$$

71. $\frac{1}{x} + \frac{1}{y} = 5$ ·

$$\frac{1}{y} + \frac{1}{z} = 6,$$

$$\frac{1}{x} + \frac{1}{z} = 7,$$

$$\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9, \text{ adding}$$

$$\frac{1}{y} + \frac{1}{z} = 6,$$

$$\frac{1}{x} = 3;\quad x=\tfrac{1}{3},\ y=\tfrac{1}{2},\ z=\tfrac{1}{4}.$$

72. x, 5 ; y, 4 ; z, 3.

73. $$\frac{2x+7}{x+2}=\frac{2y+1}{y},\quad \frac{x+3}{x-2}=\frac{3z+1}{3z-1},\quad \frac{y+3}{y+1}=\frac{z+2}{z+1}$$

$$\therefore \frac{3}{x+2}=\frac{1}{y},\quad \frac{5}{x-2}=\frac{2}{3z-1},\quad \frac{2}{y+1}=\frac{1}{z+1},$$

$$x-\ \ 3y+2=0$$

$$2x-15z+1=0$$

$$y-\ \ 2z-1=0$$

$$x=+7,\ y=+3,\ z=1.$$

74. x, 2 ; y, 3 ; z, 1.

75. Equation reduce to $\frac{3}{x} + \frac{18}{y} = 9$

$$-\frac{3}{x} + \frac{5}{z} = -2$$

$$-\frac{21}{y} - \frac{5}{z} = -8$$

$-\frac{3}{y} = -1$ adding y, 3 ; x, 1 ; z, 5.

76. x, 0 ; y, 1 : z, 2.

77. Equations reduce to $x-by+3z=0$, $5x-11y-z=0$, $21x+31y+41z=135$; from which $y=360 \div 349$, $x=1755 \div 698$.

78. First two equations become $2xy+6xz-4yz=0$, $6xy+3xz-3yz=0$, $\therefore$ $5xz-3yz=0$, or $5x=3y$; so $3y-z=0$, or $z=3y$. Substitute these values of x and z, in third equation $=x$, $\frac{1}{5}$; y, $\frac{1}{3}$; z, 1.

79. Eliminate u between second and third and between second and fourth, and add results ; $10z=10$, $z=1$; x, 5 ; y, 4 ; u, 3.

80. Eliminate u ; $5x+2y-3z=22$, $x+4y-3z=11$, $4x-2y=11$, $4x+28y=110$; $x=4\frac{2}{5}$, $y=3\frac{3}{10}$, $z=2\frac{2}{10}$, $u=\frac{11}{10}$.

81. u, 21 ; x, 31 ; y, 41 ; z, 51. 82. u, $8\frac{1}{2}$; x, 7 ; y, $4\frac{1}{2}$; z, 4.

83. u, 30 ; x, 20 ; y, 10 ; z, 0.

84. u, $\frac{1}{4}$; x, $11 \div 24$; y, $\frac{1}{4}$; z, $1 \div 24$.

85. x, $270 \div 117$; y, $-52 \div 117$; z, $15 \div 117$; u, $-126 \div 117$.

86. $u=x=y=z=210$.

Exercise lviii., page 189.

1. $y=(a'c-ac') . (a'b-ab')$. 2. $y=b(cn-dm) \div (ad-bc)$.

3. $y=b(d-c)(d-a) \div d(b-c)(b-a)$.
$z=c(d-a)(d-b) \div d(c-a)(c-b)$.

4. $y=cz+du+ew+ax$, $z=du+ew+ax+by$,
$u=ew+ax+by+cz$, $w=ax+by+cz+du$.

5. (1) .... $\frac{x}{m} + \frac{y}{n} = a$

(2) .... $\frac{y}{n} + \frac{z}{p} = b \qquad y=\frac{1}{2}x(b-c+a)$

(3) .... $\frac{x}{m} + \frac{z}{p} = c \qquad z=\frac{1}{2}p(c-a+b)$

$(1)-(2)+(3), \quad \frac{2x}{m} = a-b+c$

$x = \frac{1}{2}m(a-b+c)$, y, z, by symmetry.

6. (1) $x+ay+bz=m$
(2) $bx+y+az=n$
(3) $ax+by+z=p$

$b(3)-1 \qquad (ab-1)x+(b^2-a)y=bp-m$

$a(3)-2 \qquad (a^2-b)x+(ab-1)y=ap-n$

$\{(ab-1)^2-(b^2-a)(a^2-b)\}x=p(a^2-b)-m(ab-1)+n(b^2-a)$

$x=\{p(a^2-b)-m(ab-1)+n(b^2-a)\} \div \{a^3+b^3-3ab+1\}$,

$y=\{m(b^2-c)-n(bc-1)+p(c^2-b)\} \div \{b^3+c^3-3bc+1\}$,

$z=\{n(c^2-a)-p(ca-1)+m(a^2-c)\} \div \{c^3+a^3-3ac+1\}$.

7. (1) $x+ay=l$, $\quad 1-(a)2, \; x-abz=l-am$.
(2) $y+bz=m$, $\quad +ab(3), \; x+abcu=l-am+abn$.
(3) $z+cu=n$, $\quad -abc(4), \; x-abcdw=l-am+abn-abcp$.
(4) $u+dw=p$, $\quad +abcd(5)$.
(5) $w+ex=r$.

$x+abcdex=l-am+abn-abcp+abcdr$,

$\therefore \; x=(l-am+abn-abcp+abcdr) \div (1+abcde)$,

$y=(m-bn+bcp-bcdr+bcdel) \div (1+abcde)$.

8. $x-by-cz=0$.................. 1
$y-cz-ax=0$.................. 2
$z-ax+by=0$.................. 3

$(1-2)$, $(1+a)x-(1+b)y=0$, $(1+a)x=(1+b)y=(1+c)z$.

Dividing the terms of (1) by these equals,

$\therefore \; \frac{1}{1+a} = \frac{b}{1+b} + \frac{c}{1+c} \quad \therefore \; 1 = \frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c}$.

9. $$x-ay-az=0 \ldots\ldots\ldots\ldots\ 1$$
$$y-bz-bx=0 \ldots\ldots\ldots\ldots\ 2$$
$$z-cx-cy=0 \ldots\ldots\ldots\ldots\ 3$$

$b(3)+2$, $(1-bc)y-(b+bc)x=0$, $c(2)+3$, $(1-bc)z-(c+bc)x=0$,

$\therefore \dfrac{x}{1-bc}=\dfrac{y}{b(1+c)}=\dfrac{z}{c(1+b)}$ $\therefore$ dividing (1) by these equals,

$1-bc=ab(1+c)+ac(1+b)$, $\therefore$ $1=ab+bc+ca+2abc$.

10. $1-2$, $\therefore x-y=by-ax$, $x(1+a)=y(1+b)$,

$\therefore$ $x(1+a)=y(1+b)=z(1+c)=u(1+d)=w(1+e)$,

dividing the terms of first equation by these equals,

$$\frac{1}{1+a}=\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}+\frac{e}{1+e},$$

$$\therefore\ 1=\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}+\frac{e}{1+e},$$

Since $\dfrac{1}{1+a}=1-\dfrac{a}{1+a}$.

Exercise lix., page 192.

1. $x=(nc-bd)\div(na-bm)$; $y=(mc-ad)\div(mb-na)$.
2. $x=(nc+bd)\div(an+bm)$; $y=(mc-ad)\div(bm+an)$.
3. $x=c(n-b)\div(an-mb)$; $y=c(m-a)\div(bm-an)$.
4. $x=(b-c)a\div(b-a)$; $y=b(a-c)\div(a-b)$.

5. $\dfrac{x}{a}+\dfrac{y}{b}=1$, $\dfrac{x}{b}+\dfrac{y}{a}=1$, $\left(\dfrac{b}{a}-\dfrac{a}{b}\right)x=b-a$,

$x=ab\div(a+b)$, $y=ab\div(a+b)$.

6. $\left(\dfrac{b}{a}+\dfrac{a}{b}\right)x=b$, $x=ab^2\div(a^2+b^2)$, $y=a^2b\div(a^2+b^2)$.

7. $x=ac\div(a+b)$, $y=bc\div(a+b)$.

8. $\dfrac{a}{x}+\dfrac{b}{y}=m$, $\dfrac{b}{x}+\dfrac{a}{y}=n$, $\dfrac{a^2-b^2}{x}=am-bn$.

$x=(a^2-b^2)\div(am-bn)$, $y=(b^2-a^2)\div(bm-an)$

9. $$(a+c)x-(a-c)y=2ab \ldots\ldots\ldots\ldots\ldots\ldots 1$$
$$-(a-b)x+(a+b)y=2ac,$$
adding $(b+c)x+(b+c)y=2a(b+c)$, $x+y=2a$,
$$(a-c)x+(a-c)y=2a(a-c)$$
adding to 1, $2ax=2a(a+b-c)$, $x=a+b-c$,
$$y=b+c-a,\ z=c+a-b.$$

10. $\dfrac{x-c}{y-c} = \dfrac{a}{b} \ldots\ldots\ldots\ldots\ldots (1) \quad \therefore \dfrac{x-y}{y-c} = \dfrac{a-b}{b}.$

But $(x-y)=a-b$, $\therefore \dfrac{a-b}{y-c} = \dfrac{a-b}{b}$, $\quad \therefore y=b+c$, $x=a+c$.

11. $\dfrac{x}{y} = \dfrac{a}{b} \quad \therefore x=\dfrac{a}{b}y, \qquad \therefore \dfrac{\frac{a}{b}y+m}{y+n} = \dfrac{c}{d}$,

$\therefore\ y=b(cn-dm)\div(ad-bc)$, $x=a(cn-dm)\div(bd-ac)$.

12. $\dfrac{x+y}{y+1} = \dfrac{a+b+c}{a-b+c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

$\dfrac{y-1}{x+1} = \dfrac{a-b-c}{a+b-c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

$\therefore$ by adding numerator to denominator, &c.,

$\dfrac{x+y}{x+1} = \dfrac{2(a-c)}{a+b-c}$; divide (1) by this and multiply result

by 2, $\quad \therefore \dfrac{y-1}{y+1} = \dfrac{a^2-(b+c)^2}{2(a-c)(a-b+c)}$,

$\therefore\ y=\{3(a^2-c^2)-b(b+2a)\}\div\{(a-b)^2-c^2+4bc.\}$

13. $\dfrac{x-a+c}{y-a+b} = \dfrac{b}{c} = \dfrac{x-(a+b-c)}{y-(a-b-c)} \ldots\ldots\ldots (1)$

$\dfrac{x+c}{y+b} = \dfrac{a+b}{a+c} = \dfrac{x-(a+b-c)}{y-(a-b+c)} \ldots\ldots\ldots (2).$

From 1, $x-(a+b-c)=\dfrac{b}{c}\{y-(a-b+c)\}$,

From 2, $x-(a+b-c)=\dfrac{a+b}{a+c}\{y-(a-b+c)\}$,

$\therefore\ x-(a+b-c)=0$ and $y-(a-b+c)=0$,

i.e., $x=a+b-c,\ \ y=a-b+c$.

14. $\dfrac{x+c}{a+b}-1+\dfrac{y+b}{a+c}-1=0$ .... (1),

$\dfrac{x-b}{a-c}-1+\dfrac{y-c}{a-b}-1=0$ .... (2),

From 1, $\dfrac{x-a-b+c}{a+b}+\dfrac{y+b-a-c}{a+c}=0$,

From 2, $\dfrac{x-a-b+c}{a+b}+\dfrac{y+b-a-c}{a-b}=0$,

$\therefore (x-a-b+c)\left(\dfrac{a-b}{a+b}-\dfrac{a-c}{a+b}\right)=0.$

$\therefore\ x=a+b-c$ and $y=c+a-b$.

15. $\dfrac{x}{m-a}+\dfrac{y}{m-b}=1,\quad \dfrac{x}{n-a}+\dfrac{y}{n-b}=1,$

$x\div(m-a)(n-a)+y\div(m-b)(n-b)=0.$

$\therefore x=(m-a)(n-a)\div(b-a),$

$y=\dfrac{(m-b)(n-b)}{a-b}.$

16. $x+y+z=0$ 1,

$(b+c)x+(a+c)y+(a+b)z=0$ 2,

$bcx+acy+abz=1$ 3,

$(a+b)1-2 \qquad (a-c)x+(b-c)y=0$

$ab(1)-3 \qquad b(a-c)x+a(b-c)y=-1$

$(a-b)(a-c)x=1,\ x=\dfrac{1}{(a-b)(a-c)}$

$y=\dfrac{1}{(b-c)(b-a)},\qquad z=\dfrac{1}{(c-a)(c-b)}.$

17. $x+y+z=l$ ....(1)

$ax+by+cz=m$ ....(2)

$$\frac{x}{l-a}+\frac{y}{l-b}+\frac{z}{l-c}=1 \text{..............(3)}$$

$2-c(1)$, $\quad (a-c)x+(b-c)y=m-lc$ ....(4)

$$\left(\frac{1}{l-c}\right)(1)-(3),\ x\left(\frac{1}{l-c}-\frac{1}{l-a}\right)+y\left(\frac{1}{l-c}-\frac{1}{l-b}\right)=\frac{l}{l-c}-1$$

$$\frac{x(c-a)}{(l-c)(l-a)}+\frac{y(b-c)}{(l-c)(l-b)}=\frac{c}{(l-c)}$$

$$\frac{x(c-a)}{l-a}+\frac{y(b-c)}{l-b}=c. \text{..............(5)}$$

$(l-b)4-(5)$, $\quad \dfrac{x(c-a)(a-b)}{(l-a)(l-b)}=\dfrac{m-bc}{(l-b)}$

$x=\{(m-bc)(l-a)\}\div\{(c-a)(a-b)\}$,

$y=\{(m-ca)(l-b)\}\div\{(a-b)(b-c)\}$,

$z=$ ....

18. $\dfrac{x-a}{p}=\dfrac{y-b}{q}=\dfrac{z-c}{r}$ ....(1)

$l(x-a)+m(y-b)+n(z-c)=1$ ....(2)

From (1) $\quad x-a=\dfrac{p}{q}(y-b)$, $\quad z-c=\dfrac{r}{q}(y-b)$.

Substituting in (2),

$$\frac{lp}{q}(y-b)+m(y-b)+\frac{nr}{q}(y-b)=1,$$

$$\therefore\ (y-b)\frac{(pl+mq+nr)}{q}=1,$$

$$y-b=\frac{q}{pl+mq+nr},\quad \therefore y=\frac{q}{pl+mq+nr}+b,$$

$$z=\frac{r}{pl+mq+nr}+c,\qquad x=\frac{p}{pl+mq+nr}+a.$$

19. $\dfrac{x-a}{p} = \dfrac{y-b}{q} = \dfrac{z-c}{r}$, $lx+my+nz=1$,

$\therefore\ l(x-a)+m(y-b)+z(n-c)=1-(la+mb+nc)$, comparing with 18,

$$x-a=\frac{p\{1-(la+mb+nc)\}}{pl+mq+nr},\quad x=\frac{p\{1-(la+mb+nc)\}}{pl+mq+nr}+a,$$

$y=$ ....

20.
$$ax-by=a^2-b^2 \text{ ....}(1)$$
$$ax-cz=a^2-c^2 \text{ ....}(2)$$
$$ax+by+cz=m^2 \text{ ....}(3)$$
$$(2)+(3)\qquad 2ax+by=m^2+a^2-c^2 \text{ ....}(4)$$
$$(4)+1\qquad 3ax=m^2+2a^2-b^2-c^2,$$

$$x=\frac{m^2+2a^2-b^2-c^2}{3a},\quad y=\frac{m^2+2b^2-c^2-a^2}{3b},\ z\text{...........}$$

21. $y=a-b+c$, z and x by symmetry.

22.
$$x+y+z=0 \text{ ....}(1)$$
$$ax+by+cz=ab+cb+ca \text{ ....}(2)$$
$$(b-c)x+(c-a)y+(a-b)z=0 \text{ ......}(3)$$
$$2-c(1)\qquad (a-c)x+(b-c)y=ab+bc+ca,$$
$$3-(a-b)\ (1),\quad (2b-a-c)x+(b+c-2a)y=0,$$

$$x\ \frac{(a-c)(b+c-2a)+(b-c)(2b-a-c)}{(b+c-2a)(2b-a-c)}=ab+bc+ca$$

$$x=(ab+bc+ca)\frac{(b+c-2a)(2b-a-c)}{(a-c)(b+c-2a)+(b-c)(2b-a-c)}$$

$$y=(ab+bc+ca)\frac{(b+c-2a)(2b-a-c)}{(a-c)(b+c-2a)+(b-c)(2b-a-c)}.$$

NOTE.—First equation should be $x+y+y+z=a+b+c$, then $x=\frac{1}{2}(b+c)$, &c.

23. $x+y+z=m$, $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, $y=\dfrac{b}{a}.x$, $z=\dfrac{c}{a}x$,

$$\frac{x(a+b+c)}{a}=m,\ \therefore\ x=\frac{ma}{a+b+c},\ y=\frac{mb}{a+b+c}.$$

24. $ax+by+cz=r, \qquad mx=ny, \qquad qy=pz,$

$$\therefore\ y = \frac{m}{n}x, \quad z = \frac{q}{p}y = \frac{mq}{pn}x,$$

$$ax + \frac{mb}{n}x + \frac{mqc}{np}x = r, \qquad \therefore\ x=(npr)\div(anp+bmp+cmq),$$

$$y=(mpr)\div(anp+bmp+cmq,) \quad z=(mqr)\div(anp+bmp+cmq).$$

25. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1)$

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{bc}{x} + \frac{ac}{y} + \frac{ab}{z} + (a-b)(b-c)(c-a) = 0 \ldots\ldots\ldots(3)$$

$2-c(1) \qquad \dfrac{a-c}{x} + \dfrac{b-c}{y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

$ab(1)-3 \qquad \dfrac{b(a-c)}{x} + \dfrac{a(b-c)}{y} = (a-b)(b-c)(c-a) \ldots\ldots(5)$

$a(4)-5 \qquad \dfrac{(b-a)(a-c)}{x} = (a-b)(b-c)(c-a), \quad \therefore\ x = \dfrac{1}{b-c},$

y, z, by symmetry.

26. $x=\frac{1}{2}(b+c-a)$, y and z by symmetry.

27. From second set of equations,

$$x = \frac{a}{d}u, \ y = \frac{b}{d}u, \ z = \frac{c}{d}u,$$

Substituting, $\dfrac{ma+nb+pc+qd}{d}x = r.$

$$x = \frac{dr}{ma+nb+pc+qd}.$$

28. $\dfrac{x(y+z)}{a} = \dfrac{y(z+x)}{b} = \dfrac{z(x+y)}{c} \ldots\ldots\ldots\ldots\ldots\ldots(1),$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a+b+c \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

Divide (1) by xyz, $\dfrac{\frac{1}{z}+\frac{1}{y}}{a} = \dfrac{\frac{1}{x}+\frac{1}{z}}{b} = \dfrac{\frac{1}{y}=\frac{1}{x}}{c} =$

$$\frac{2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}{a+b+c} = 2, \quad \therefore \frac{1}{z}+\frac{1}{y} = 2a,$$

$$\frac{1}{x}+\frac{1}{z} = 2b, \quad \frac{1}{y}+\frac{1}{x} = 2c, \quad \frac{1}{x}-\frac{1}{y} = 2(b-a),$$

$$\frac{1}{x}+\frac{1}{y} = a, \quad \frac{2}{x} = 2(b+c-a), \; x = 1 \div (b+c-a.)$$

$\therefore \; (a-b)(x+c)-a(y+b)+b(z+a)=0$, and

$a(x+c)+(c-a)(y+b)-c(z+a)=0$,

$\therefore \; \{(a-b)c+ab\}(x+c)+\{(c-a)b-ac\}(y+b)=0$,

$\therefore \; (ac-bc+ab)\{(x+c)-(y+b)\}=0$,

$\therefore \; x+c=y+b=$ (by symmetry) $z+a$, $\therefore$ since $x+y+z=$ $2(a+b+c)$, $x=a+b$, $y=c+a$, $z=b+c$.

30. Adding, $2(ax+by+cz)=3$, $ax+by+cz=\frac{3}{2}$.

Subtracting first equation,

$$cz=\tfrac{1}{2}, \; z=1\div 2c, \; x=1\div 2a, \; y=1\div 2b.$$

31. Multiplying (1) by n, (2) by m, (3) by l,

$$lny+mnx=n^2, \qquad mnx+lmz=m^2 \qquad lmz+lny=l^2.$$

Adding, $mnx+lny+lmz=\dfrac{l^2+m^2-n^2}{2}$.

Subtracting first equation, $lmz = \dfrac{l^2+m^2-n^2}{2}$,

$$z = \frac{l^2+m^2-n^2}{2lm}, \qquad x = \frac{m^2+n^2-l^2}{2mn}, \qquad y = \frac{n^2+l^2-m^2}{2nl}.$$

32. Adding, $x+y+z = \dfrac{a+b+c}{2}$.

Subtracting second equation, $x = \dfrac{a+c-b}{2}$.

33. Adding, $x+y+z = \frac{mn}{l} + \frac{ln}{m} + \frac{lm}{n}$.

Subtracting first equation, $2x = \frac{ln}{m} + \frac{lm}{n}$,

$$x = l.\frac{m^2+n^2}{2mn}, \quad y = m.\frac{n^2+l^2}{2nl}.$$

34. Adding, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = (a+b+c)$

Subtracting first equation, $\frac{1}{x} = b+c-a$,

$$x = \frac{1}{b+c-a}, \quad y = \frac{1}{c+a-b}.$$

35. Adding, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}$.

Subtracting first equation, $\frac{2}{x} = \frac{2}{b} + \frac{2}{c}$,

$$x = \frac{bc}{b+c}, \quad y = \frac{ca}{c+a}.$$

36. $x=b+c-a$; y, z, by symmetry.

37. x, a; y, b; z, c.

38. Adding $\frac{x}{b+c} + \frac{y}{c+a} + \frac{z}{a+b} = 0$.

Subtracting first $=n$, $\frac{z}{a+b} = a-b$,

z, a^2-b^2; x, b^2-c^2; y, c^2-a^2.

39. $x=\frac{1}{5}(a+2b-c+3d)$; y, z, v, by symmetry

40. $u=\frac{1}{11}(4a+b+3c-2d+5e)$, x, &c., by symmetry.

Exercise lx., page 202.

1. Add the equations, $\therefore$ $2a(x+y)=4a^2$, subtract, $\therefore$ $2b(x-y)=4b^2$; $x=a+b$, $y=a-b$.

2. $x+y=a$, $x-y=\frac{b}{a}$, $x=\frac{1}{2}(a^2+b)\div a$, $y=\frac{1}{2}(a^2-b)\div a$.

3. Factoring in second equation $(x+y)(2x-3y)=n^2$, $\therefore$ from first, $x+y=n^2m$; $x=(3n^2+m^2)\div 5m$, $y=(2n^2-m^2)\div 5m$.

4. Dividing first $=n$ through by a^2-b^2, $\therefore \dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{1}{a-b}$; add to second equation, $\therefore 2x\div(a+b)=2a\div(a^2-b^2)$;

$$x, \frac{a}{a-b}; \quad y, \frac{b}{a+b}.$$

5. From 1st equation, $(a^2-b^2)x+(a+b)y=a+b+1$; add second equation, $\therefore (a^2-b^2-1)x=(a^2-b^2-1)\div(a-b)$;
$x=1\div(a-b)$; y, $1\div(a+b)$.

6. Substitute in first equation the value of x from second.
$\therefore \{(a+b-c)^2-(a-b+c)^2\}y=4a(b-c)(a-b+c)$
or, $4a(b-c)y=4a(b-c)(a-b+c)$,
$\therefore y=a-b+c, \qquad x=a+b-c.$

7. From first equation, $x=(a+b-c)y\div(a-b+c)$.
Substitute this value of x in second equation and we have
$\{b(a+b)-c(a+c)\}y=(a-b+c)\{b(a+b)-c(a+c)\}$,
$\therefore y=a-b+c, \qquad x=a+b-c.$

8. First equation gives $(a+b)x-(a-b)y=2ab$; second gives $(a^3+b^3)x-(a^3-b^3)y=0$; $x=(a^2+ab+b^2)\div(a+b)$;
y, $(a^2-ab+b^2)\div(a-b)$.

9. From the first, $x-y=(a+1)\div(a-1)$, and from second $x+y=(b+1)\div(b-1)$; $x=(ab-1)\div(a-1)(b-1)$;
y, $(a-b)\div(a-1)(b-1)$.

10. From first, $(x+1)\div y=a$, or $x=ay-1$; and from second $x\div(y+1)=1\div b$, $\therefore (ay-1)(y+1)=1\div b$;
$x=(1+a)\div(ab-1)$, y, $(1+b)\div(ab-1)$.

11. From first, $x\div(1-y)=(a+1)\div(a-1)$; from second $x\div(1+y)=(b+1)\div(b-1)$; $x=(a+1)(b+1)$,
$\div(ab-1)$, $y=(a-b)\div(ab-1)$.

12. x, $a(a+b)$; y, $b(a-b)$.

13. $x=a\{b(a+b)-c(a-c)\}\div(a^2-bc)$;
$y=a\{b(a-b)+c(a+c)\}\div(a^2-bc)$.

14. $x=-(a+b)$, $y=ab$.

15. Adding the equations and subtracting each separately from sum, $x=\frac{1}{2}(b+c)$, $y=\frac{1}{2}(c+a)$, $z=\frac{1}{2}(a+b)$.

16. Add all the equations, $\therefore$ $19(x+y+z)=(a+b+c)$;
$x=(a-2b+3c)\div 38$.

17. Adding the three equations and subtracting each $=n$, separately, $x=1\div(b+c)$; y, $1\div(c+a)$; z, $1\div(a+b)$.

18. See question 29, Exercise 59.

19. $y=(c+a)(c-a)$, $z=(a+b)(a-b)$, $x=(b+c)(b-c)$.

20. Subtracting first equation from third,
$z=a^2-b^2$; x, b^2-c^2; y, c^2-a^2.

21. Proceeding as in example 7, page 194, assume

$1-\dfrac{x}{t}-\dfrac{y}{t-1}-\dfrac{z}{t-2}=(t^3+Bt^2+ct+D)\div t(t-1)(t-2)$; but from the equations it is seen that the left-hand member of this equation vanishes for $t=a$, or b, or c, $\therefore$ $t^3+Bt^2+ct+D=(t-a)(t-b)(t-c)$; multiply equation by t and put $t=0$ in result, $\therefore$ $x=\frac{1}{2}abc$; similarly multiply by $t-1$ and put $t-1=0$, $\therefore$ $y=(1-a)(1-b)(1-c)$; Again, multiply by $t-2$ and put this $=0$, $\therefore z=(2-a)(2-b)(2-c)$.

22. Equations are $xy=a(x+y)$, &c, divide the first by axy, the second by byz, the third by czx, $\therefore$

$$\frac{1}{a}=\frac{1}{x}+\frac{1}{y},\quad \frac{1}{b}=\frac{1}{y}+\frac{1}{z},\quad \frac{1}{c}=\frac{1}{z}+\frac{1}{x},$$

subtract the second of these from the first and add result to third,

$\therefore \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{2}{x}$ $\therefore$ $x=2abc\div(ab+bc-ca)$, y and z by symmetry.

23. $\dfrac{x-1}{a}+\dfrac{y-1}{b}+\dfrac{z-1}{c}=0,$

$\dfrac{x-1}{b}+\dfrac{y-1}{c}+\dfrac{z-1}{b}=0, \therefore x-1=y-1=z-1,$

$\dfrac{x-1}{c}+\dfrac{y-1}{a}+\dfrac{z-1}{b}=0, \therefore x=y=z=1.$

24. $\dfrac{x}{a}=\dfrac{mx}{ma}=\dfrac{ny}{nb}=\dfrac{pz}{pc}=\dfrac{qu}{qd}=(mx+ny+pz+qu)\div$ $(ma+nb+pc+qd)=r\div(ma+\ldots\ldots\ldots)$. $\therefore x=$ $ar\div(ma+nb+pc+qd)$, $y=br\div(\ldots\ldots\ldots)$, $z=$ &c.

25. Substitute in last equation the values of u derived from the others, $\therefore u^2\left(\dfrac{d^2}{b^2}-\dfrac{d^2}{c^2}\right)=\left(\dfrac{d}{a}-1\right)u, \quad \therefore u=0$, or

$$=\left(\frac{d}{a}-1\right)\div\left(\frac{d^2}{b^2}-\frac{d^2}{c^2}\right).$$

26. $x=(b+c-a)\div(a+b+c)$; $y=(b-a-c)x\div(a-b-c)$.

27. $x=\frac{1}{2}(a-b+m-n)$; z, $\frac{1}{2}(a-b-m+n)$.

28. $x=(4a+2c-d-3b)\div 40$, y, z, u by symmetry.

29. As in Ex. 3, page 193, the polynome $t^4+ut^3+zt^2+yt+x$ vanishes for $t=a$, $t=b$, $t=c$, and $t=d$, and $\therefore =$

$(t-a)(t-b)(t-c)(t-d)=t^4-(a+b+c+d)t^3+$
$(ab+ad+ac+bd+bc+cd)t^2-(abc+abd+acd+bcd)t+abcd,$
$\therefore a=-(a+b+c+d),\ z=(ab+\&c.),$
$y=-(abd+abc+bcd+acd),\ x=abcd.$

30. $x=\frac{1}{2}(a-b+c-d+e)$; y, z, u, v, by symmetry.

31. Solve as 8, Exercise LVIII., $x=(a-lb+lmc-lmnd+lmnpc)\div(1+lmnpq)$, whence y, z, u, v, by symmetry.

33. $x=b+c-e$, $y=+ed-a$, &c.

34. $(1)+5(2)+3(3)-7(4)+9(5)$, $\therefore 22y=a+5b+3c-7d+9e$, and the values of the others may be written down by symmetry.

35. $2u-2y=d-e$, $2z-2x=c-d$, $2z+2u-2x=b+c$, $2u-2a=a-b$, $\therefore$ $2z=a+c$, $2a=b+d$, &c.

36. $5+2(4)+(3)-(4)$, $\therefore$ $3z=3c+3d+3e$, $z=c+d+e$, and u, v, x, y, by symmetry.

37. $x=a-2b+3c-2d+e$; and y, z, u, v, by symmetry.

Exercise lxi., page 204.

1. $\{(1+x+x^2)+(1+y+y^2)\}\div\{(1+x+x^2)-(1+y+y^2)\}=(a+1)\div(a-1)$; so, $(1+x+x^2+1+y+y^2)\div\{(1+y+x^2)-(1+x+y^2)\}=(b+1)\div(b-1)$,

$\therefore$ $(x+y-1)\div(x+y+1)=(a+1)(b-1)\div(a-1)(b+1)$, and $x+y=\{(a+1)(b-1)+(a-1)(b+1)\}\div\{(a-1)(b+1)-(a+1)(b-1)\}=(ab-1)\div(a-b)$,

$\therefore$ $y=(ab-1)\div(a-b)-x$, substitute in (1); $x=(2ab+a+b+r)\div 2(a-b)$, where $r^2=4a(b^2+b+1)+(3a-b)(3b-a)$.

2. From (1), $(x^2+x+1)\div(x^2-2x+1)b^2=(y^2+y+1)\div(y-1)^2$ $\therefore$ $3x\div3(x^2+1)=\{b^2(y^2+y+1)-(y-1)^2\}\div\{2b^2y^2+2b^2y+2b^2+(y-1)^2\}$, $\therefore$ $(x+1)^2\div(x-1)^2=\{4b^2(y^2+y+1)-(y-1)^2\}\div3(y-1)^2$; from (1), $(x+1)^2\div(x-1)^2=a^2(y+1)^2\div(y-1)^2$. $\therefore$ this right-hand member $=\{4b^2(y^2+y+1)-(y-1)^2\}\div3(y-1)^2$, where $y-1$ cannot be zero if x is finite, $\therefore 3a^2(y+1)^2=4b^2(y^2+y+1)-(y-1)^2$, &c., $x=(ar+1)\div(ar-1)$, where $r^2=(b^2-1)\div3(a^2-b^2)$.

3. Multiply the two equations together, member by member, $\therefore$ $(1+x)^2\div(1-x)^2=(1+a)(1+b)\div(1-a)(1-b)$. Taking sq. root, and then sum and difference of numerator and denominator, $\therefore x=\{\sqrt{(1+a)(1+b)}-\sqrt{(1-a)(1-b)}\}\div\{\sqrt{(1+a)(1+b)}+\sqrt{(1-a)(1-b)}\}$; $y=\{\sqrt{(1+a)(1-b)}-\sqrt{(1-a)(1+b)}\}\div\{(1+a)(1-b)+\sqrt{(1-a)(1+b)}\}$.

4. Applying (6), page 122, we have from first equation,

$$\frac{x+y+xy+1}{xy+1-x-y} = \frac{(x+1)(y+1)}{(x-1)(y-1)} = \frac{a^2}{\alpha^2}:$$

Similarly from second equation, $\frac{(x+1)(y-1)}{(x-1)(y+1)} = \frac{b^2}{\beta^2}$.

Multiply these results together and take square root,

$$\therefore\ (x+1) \div (x-1) = ab \div \alpha\beta.$$

And again applying (6), page 122, $x=(ab-\alpha\beta) \div (ab+\alpha\beta)$; *dividing* the results we get $y=(a\beta+b\alpha) \div (a\beta-b\alpha)$.

5. Proceeding exactly as in last example we have

$(x+1)^2 \div (x-1)^2 = (a+b+c)(a+b-c) \div (b+c-a)(a+c-b)$,

$\therefore\ x=\{\sqrt{(a+b+c)(a+b-c)}+\sqrt{(b+c-a)(a+c-b)}\} \div$
$\{\sqrt{(a+b+c)(a+b-c)}-\sqrt{\ldots\ldots\ldots}\}$.

Note.—Rationalizing the denominator of this, the value of x appears in the form $(2ab-4A) \div (a^2+b^2-c^2)$, where $16A^2=$ the product of the *four* factors involved.

6. $x=(a+b)\div(1-ab)$.

7. $x=(\alpha\beta-ab)\div(a\beta+b\alpha)$.

8. $\frac{(1+xy)^2+(x+y)^2}{2(x+y)(1+xy)} = \frac{2a}{2m} = \frac{a}{m}$; now proceed as in examples 4 and 5 above, $\therefore\ \{(1+xy)+(x+y)\}^2 \div \{1+xy-(x+y)\}^2$ or $\{(x+1)(y+1)\}^2 \div \{(x-1)(y-1)\}^2=(a+m)(a-m)$.

Similarly the second equation gives

$$\{(x+1)(y-1)\}^2 \div \{(x-1)(y+1)\}^3.$$

Multiplying these results, $\therefore\ \left(\frac{x+1}{x-1}\right)^4 = \left\{\frac{(a+m)(b+n)}{(a-m)(b-n)}\right\}$,

$$\therefore = \frac{\{(a+m)(b+n)\}^{\frac{1}{4}}-\{a-m)(b-n)^{\frac{1}{4}}\}}{\ldots\ldots\ldots\ +\ \ldots\ldots\ldots}$$

9 $x=\{a\sqrt{(1-b^2)}-b\sqrt{(1-a^2)}\}\div\sqrt{(a^2-b^2)}$. Proceed exactly as in last example.

10, 11, 12. See examples 14, 13, 12, page 197, Hand-Book.

13, 14. See example 15, page 198, and example 11, page 196.

15. $x=a$, $y=b$, $(2)\div(1)$, $\dfrac{x^2+x+1}{y^2+y+1} = \dfrac{a^2+a+1}{b^2+b+1}$, in which substitute the value of y from 1.

16. $a\div b=(x^2+y^2)\div xy$, $\therefore$ $(a+b)\div(a-b)=(x^2+xy+y^2)\div(x^2-xy+y^2)=xy\div b=(x^2+y^2)\div a$, $\therefore$ $xy=b\times\dfrac{a+b}{a-b}$, $x^2+y^2=a\times\dfrac{a+b}{a-b}$, $\therefore$ $(x+y)^2=\dfrac{a+b}{a-b}(a+2b)$ and

$(x-y)^2=\dfrac{a+b}{a-b}(a-2b)$, $\therefore$ $2x=$ &c.

17. $x^2+xy+y^2=b$, and by division, $x^2-xy+y^2=\dfrac{b}{a}$, $\therefore$ $x^2+y^2=\frac{1}{2}\left(b+\dfrac{a}{b}\right)$, $xy=\frac{1}{2}\left(b-\dfrac{a}{b}\right)$, $\therefore$

$(x+y)^2=\frac{1}{2}\left(b+\dfrac{a}{b}\right)+b-\dfrac{a}{b}=\frac{1}{2}\left(3b-\dfrac{a}{b}\right)$, and

$(x-y)^2=\frac{1}{2}\left(b+\dfrac{a}{b}\right)-b+\dfrac{a}{b}=\frac{1}{2}\left(\dfrac{3a}{b}-b\right)$, $\therefore$ &c.

18. $(x^2-y^2)(x^2-xy+y^2)=a$, $xy(x^2-y^2)=b$, $\therefore$ $(x^2-xy+y^2)\div xy=a\div b$, $\therefore$ $(x+y)^2\div(x-y)^2=(a+3b)\div(a-b)$, $\therefore$ $(x+y)\div(x-y)=\sqrt{(a+3b)}\div\sqrt{(a-b)}=m$, suppose, find x from this and substitute in second equation, getting a quadratic in y^2.

19. Add the equations, $\therefore$ $2xy=(a+b)x^2-(a-b)y^2$. Put $x=vy$, $\therefore$ $2vy^2=(a+b)v^2y^2-(a-b)y^2$, or $2v=(a+b)v^2-(a-b)$, $\therefore$ $v=\dfrac{1}{a+b}\pm\dfrac{1}{a+b}\sqrt{(a^2-b^2+1)}=m$ suppose. Substitute in first equation.

20. Subtract second from first, $\therefore$ $x^3-y^3=(a-b)x^2+(a-b)y^2+(a-b)xy=(a-b)(x^2+xy+y^2)$ $\therefore$ $x-y=a-b$; similarly, add the two equations, and we find $x+y=a+b$; $\therefore$ $x=a$, $y=b$.

21. Add (1) and (2), $4c(x^2+y^2)=2a(x-y)^2$; $x^2+y^2 \div (x-y)^2 = a \div 2c$, $\therefore$ $(x^2+y^2)\div 2xy = a \div (a-2c)$, $\therefore$ $(x+y)^2 \div (x-y)^2 = (a-c)\div c$, and $(x+y)\div(x-y) = \sqrt{(a-c)} \div \sqrt{c}$; $x \div y = \{\sqrt{(a-c)}+\sqrt{c}\} \div \{\sqrt{(a-c)}-\sqrt{c}\}$. $=m$ suppose, $\therefore$ $x=my$, which substitute in (2), &c.

22. Subtract second from first, and $x^3-y^3 = \dfrac{c}{a}(x^2+xy+y^2)(x+y)$. If $x^2+xy+y^2=0$, then from (1) and (2) $x=y=u$, or $=\frac{1}{2}(-1\pm\sqrt{-3})u$; if x^2+xy+y^2 is not zero, divide the above result by it, $\therefore$ $x-y = \dfrac{c}{a}(x+y)$, or $(x-y)\div(x+y) = c \div a$, $\therefore$ $x \div y = (a+c)\div(a-c)$, or $x = \left(\dfrac{a+c}{a-c}\right) y = my$ suppose; then $y^3-u^3 = \dfrac{b-c}{2a}\{(x+y)^3 - xy(x+y)\} = y^3\left(\dfrac{b-c}{2a}\right)\{m+1)^3 - m(m+1)\}$, which is of the form $py^3=r$.

23. From (1) $(x+x^2+y+y^2)\div(x-y)(x+y+1) = (a+1)\div(a-1)$.........(3). From (2) $(x-y)(x+y-1)\div(x+x^2+y+y^2) = (b-1)\div(b+1)$.........(4). (3) $\times$ (4) $\therefore$ $(x+y+1)\div(x+y-1) = (a+1)(b-1)\div(a-1)(b+1)$, $\therefore$ $x+y=\{(a+1)(b-1)+(a-1)(b+1)\} \div \{(a-1)(b+1)-(a+1)(b-1)\} = (ab-1)\div a-b$, $\therefore$ $y=(ab-1)\div(a-b)-x$; substitute this value of y in either equation.

24. $\therefore$ $(x+y)^2 \div (x-y)^2 = (a+2\div(a-2)$, or $x+y = \sqrt{(a+2)} \div \sqrt{(a-2)}$, $\therefore$ $x\div y = \{\sqrt{(a+2)}+\sqrt{(a-2)}\} \div \{\sqrt{(a+2)}-\sqrt{(a-2)}\} = p$ suppose. Similarly, $(1+xy)\div(1-xy) = \sqrt{(b+2)}\div\sqrt{(b-2)}$, $\therefore xy = \{\sqrt{(b+2)}-\sqrt{(b-2)}\} \div \{\sqrt{(b+2)}-\sqrt{(b-2)}\} = m$ suppose;

$$xy \times \frac{x}{y} = x^2 = pm,\ x = \pm\sqrt{(pm)}.$$

25. Add the equations, $xy+yz+zx=\frac{1}{2}(a+b+c)$,

$\therefore\ xy=\frac{1}{2}(a+b-c)$, yz, zx, by symmetry; multiply these,

$x^2y^2z^2=\frac{1}{8}(a+b-c)(b+c-a)(c+a-b)$, from which xyz is found; divide this by $xy=$ &c. $yz=$ &c., $zx=$ &c.

26. See example 1, page 187.

27. $\therefore\ xy+yz+zx=a-x^2=b-y^2=c-z^2=v$ suppose, then $x=\sqrt{(a-v)}$, $y=\sqrt{(b-v)}$, $z=\sqrt{(c-v)}$, substituting these values in $xy+yz+zx=v$ we get

$$\sqrt{(a-v)(b-v)}+\sqrt{(b-v)(c-v)}+\sqrt{(c-v)(a-v)}=v,$$

$\therefore\ \{\sqrt{(a-v)(b-v)}+\sqrt{(b-v)(c-v)}\}^2=\{v-\sqrt{(c-v)(a-v)}\}^2$, or $2\sqrt{(c-v)(a-v)}=ca-ab-bc+2bv$, $\therefore\ 4ca-4(c+a)v+4v^2=(ca-ab-bc)^2+4b(ca-ab-bc)v+4b^2v^2$, a quadratic from which v may be found $=\pm m$ suppose, $\therefore\ a-x^2=\pm m$, &c.

28. $(z+x-y)(x+y-z)=a$, $(y+z-x)(x+y-z)=b$, $(y+z-x)(z+x-y)=c$; multiply together and extract root, $\therefore$ $(x+y-z)(y+z-x)(z+x-y)=\sqrt{abc}$, $\therefore\ x+y-z=\sqrt{(abc)}\div c$, $y+z-x=\sqrt{(abc)}\div a$, $z+x-y=\sqrt{(abc)}\div c$. Add last three equations, $\therefore\ x+y+z=\sqrt{(abc)}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$,

$$\therefore\ x=\tfrac{1}{2}\sqrt{(abc)}\left(\frac{1}{b}+\frac{1}{b}\right),\ y \text{ and } z \text{ by symmetry.}$$

29. $x^2+y^2=az$; from (2) and (3), $x^2+y^2=\frac{1}{4}(b+c)^2z^2+\frac{1}{4}(b-c)^2z^2$, $\therefore\ \frac{1}{4}z^2\{(b+c)^2+(b-c)^2\}=az$, and $z=0$, or $2a\div(b^2+c^2)$; if $z=0$, then $x=y=0$; if $z=2a\div(b^2+c^2)$ then $x=\frac{1}{2}(b+c)\times 2a\div(b^2+c^2)=a(b+c)\div(b^2+c^2)$, and $y=a(b-c)\div(b^2+c^2)$.

30. From (1) and (2), $1\div x^2=(a+b)\div z^2$, and $1\div y^2=(a-b)\div z^2$, $\therefore\ 1\div x+1\div y=\{\sqrt{(a+b)}+\sqrt{(a-b)}\}\div z=1\div c$, $\therefore\ z=c\{\sqrt{(a+b)}+\sqrt{(a-b)}\}$, $x=c\{\sqrt{(a+b)}+\sqrt{(a-b)}\}\div\sqrt{(a+b)}$, $y=c\{\sqrt{(a+b)}+\sqrt{(a-b)}\}\div\sqrt{(a-b)}$.

31. From (2) and 3), $x^3 - y^2 = bcz^2$, $\therefore$ $bcz^2 = az$. $z = 0$, or $a \div bc$; if $z = 0$, then $x = y = 0$; if $z = a \div bc$, then $x = a(b+c) \div 2bc$, and $y = a(b-c) \div 2bc$.

32. $(x+y)(z+1) = 2bz \div a$, $\therefore$ $x + y = 2bz \div a(z+1)$, $x - y = 2a \div (z+1)$, $\therefore$ $x = (bz + a^2) \div a(z+1)$, $y = (bz - a^2) \div a(z+1)$, $\therefore$ $xy = (b^2z^2 - a^4) \div a^2(z+1)^2 = (z-1) \div (z+1)$, $\therefore$ $z = -1$, or $b^2z^2 - a^4 = a^2(z^2 - 1)$, and $z^2(a^2 - b^2) = a^2(a^2 - 1)$,
$\therefore$ $z = a\sqrt{(a^2 - 1)} \div \sqrt{(a^2 - b^2)}$.

CHAPTER VII.

PAPER I., page 207.

1. Add coefficients of x, $(a+b+c)x$, and by symmetry, the same coefficients for y and z, $\therefore$ $(a+b+c)(x+y+z)$.

2. (1) $(x+2y)^3(x-2y)^3=(x^2-4y^2)^3$, &c.

(2) $2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4$.

3. (1) $x^4-6x^3+13x^2-12x+4$.

(2) Factors of dividend are x^2+3+x^{-2}, and x^2-3+x^{-2} $\therefore$ quotient $=$ &c.

(3) $x^{n^2-n}+x^{n^2-2n}+x^{n^2-3n}+$ &c.

$$= x^{n(n-1)}+x^{n(n-2)}+x^{n(n-3)}, \text{ \&c.}$$

4. (1) $2x^{2m}-\frac{1}{3}x^{2m}$. (2) $\frac{a}{b}-\frac{b}{c}-\frac{c}{a}$.

5. Cube by formula [6], and for the *sum* of the cubes substitute its value (3) from the given equation.

$\therefore$ $2+3\sqrt[3]{(1-4x^2)}\times 3=27$, $\therefore$ $1-4x^2=15625\div 729$,

$\therefore$ $x=\pm 14\sqrt{(-19)}\div 27$.

(2) $\therefore$ $-\dfrac{4}{x+2}+\dfrac{4}{x-2}=\dfrac{12}{x-3}$,

$\therefore$ $4\div(x^2-4)=3\div(x-3)$, $\therefore$ $3x^2-4x=0$, $x=0$, or $1\frac{1}{3}$.

6. $x=$ number of oxen, $320\div x=$ cost of each, $320\div(x+4)$ $=$ supposed cost; difference of these costs $=4$, $\therefore$ $x^2+4x-320$ $=0$, or $(x+20)(x-16)=0$, $\therefore$ $x=-20$, or 16. The negative result indicates a problem allied to the given one, the words *more* and *less* being merely interchanged.

7. (1) Divide numerator of first by b^2, and denominator by d^2, and substitute for $\frac{c}{d}$; result is $b^2 \div d^2$; do same in second ratio, result is $b^2 \div d^2$, $\therefore$ &c. (2). 199.

8. $AB+BC=82$, $BC+CA=97$, $CA+AB=89$. Subtract first from second, $\therefore$ $CA-AB=15$. Add result to third.

$\therefore$ $2AB=74$, and $AB=37$. $CA=52$, $BC=45$.

PAPER II., page 208.

1. $\{(a+b+c)-c\}^3=(a+b)^3$.

2. Factor first quantity, $(a+b)^3+c^3-3ab(a+b+c)=$ $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$; second quantity $=$ $(a+b+c)^2$, $\therefore$ $a+b+c$. 3. $1\div x^3$.

4. (1) Numerator $=a^{3m}-1+a^{2m}-1=(a^m-1)$ $(a^{2m}+a^m+a^m+1)$;

Denominator $=a^{2m}-a+a^m-1=(a^m-1)(a^m+1+1)$, $\therefore$ &c.

(2) Numerator $=(a+b+c)^2$; denominator $=a^2-(b+c)^2=$ $(a+b+c)(a-b-c)$, $\therefore$ $(a+b+a)\div(a-b-c)$.

5. (1) Given function of x divided by $x-a$, gives remainder (found by substituting a for x), a^3-pa^2+qa-r, which, by the question, $=0$, $\therefore$ the given function is exactly divisible by $x-a$.

(2) Put $a=0$, the expression becomes $(b+c)bc-(b+c)bc=0$, $\therefore$ a is a factor, and by symmetry b and c are factors, $\therefore$ expression $=nabc$; put $a=b=c=1$ and n is found $=1$.

6. Given expression $=x^{\frac{1}{n}}(x^{\frac{1}{m}}\div x^{\frac{1}{n}}+1)=x^{\frac{1}{n}}\left\{\left(\frac{a+b}{a-b}\right)^2+1\right\}$

$=x^{\frac{1}{n}}\left\{\frac{2(a^2+b^2)}{(a-b)^2}\right\}$, multiply this by $\frac{1}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)$, and it becomes $x^{\frac{1}{n}}\left\{\frac{a+b}{a-b}\right\}$ which $=\left\{\frac{a+b}{a-b}\right\}^{\frac{2m}{n-m}}\times\frac{a+b}{a-b}=\left\{\frac{a+b}{a-b}\right\}^{\frac{m+n}{n-m}}$.

7. (1) Complete the divisions and the equation reduces to $8x-16=16x-9$; $x=-\frac{7}{8}$.

(2) Equation reduces to $2\div(x^2+6x+8)=25\div(12x^2+74x+112)$, $\therefore x=4$, or 6.

8. Let $x=$ increase, then $r\div(p+x)=r\div p-q$; $x=p^2q\div(r-pq)$.

9. Let $x=$ third digit, $\therefore 2x=$ second, $9-x=$ first, $\therefore 9+2x=17$, $x=4$; 584.

10. First by common rule, let $x=$ equated time, then $(a+b)x=am+bn$. Second, suppose $m>n$; the *interest* of \$$b$ for the time $x-n$, must equal the *discount* of \$$a$ for the time $m-x$, or $b(x-n)\cdot 05=a(m-x)\cdot 05\div\{1+(m-x).05\}$, a quadratic for determing x.

Paper III., page 209.

1. (1) $(x+y+z)(x-y-z)\times\dfrac{x+y-z}{x+y+z}=(x-z-y)(x-z+y)$
$=(x-z)^2-y^2=-117$. (2) $a^2(z-x)+(x-y)ab+(y-z)b^2$.

2. $a+bx+cx^2$, which may be found by factoring, as in Art. XXIII. $3-4x+7x^2-10x^3=$ square root.

3. (1) See Algebra, page 145. Equation reduces to form

$$\frac{1}{x+1}+\frac{1}{x+4}=\frac{1}{x+2}+\frac{1}{x+3},\quad\therefore x=2\tfrac{1}{2}.$$

(2) $x=6$, $y=9$, $z=12$.

4. Let $x=$ number of dozen bought, $\therefore 45x=$ price paid,

5. Let $x=$ number of bushels in lower priced, $\therefore 75-x=$ do. in higher.

$\therefore 40(x+\frac{5}{3})=45x$, $\therefore x=\frac{200}{15}$ dozen $=160$ oranges.

$\dfrac{78\ 75}{75}=\$1.05$ lower priced per bushel.

$\dfrac{90}{75}=\$1.20$ higher " "

$\therefore 105x=(75-x)120$, $\therefore x=40$; 35.

6. $\sqrt{\frac{1}{2}(1+a)}+\sqrt{\frac{1}{2}(1-a)}$.

7. Equation reduces to $(40x-101)(x-3)=22(2x-7)(x-2)$, whence $x=5$, or $\frac{1}{4}$. Since the given equation has equal roots, $ax^2-36x+81$ is a complete square, *i.e.*, $4a\times 81=(36)^2$, $\therefore\ a=4$.

8. See Algebra, page 124.

9. Substitute $-(b+c)$ for a, $-(c+a)$ for b, and $-(a+b)$ for c, in the given expression, and it becomes

$$-a^2(b^2-c^2)-b^2(c^2-a^2)-c^2(a^2-b^2),\ \text{which}=0,$$

$\therefore\ a+b+c$ is a divisor, &c. See Algebra, page 43, prob. 12.

PAPER IV., page 210.

1. $\{(a+b)^2-c^2\}\{(a+c)^2-b^2\}\div\{a^2-(b+c)^2\}=(a+b+c)^2$.

2. The first and second fractions requiring to be inverted *once* and *three* times respectively, will finally stand inverted, while the third, requiring *two* inversions, will stand unchanged. $\therefore$ result $=a\div b$.

3. (1) $(2x-3y)(2x+3y)$, $(2x-3y)(2x-2y)$, $(2y-3y)(3x-2y)$.
$\therefore$ L. C. M. $=(2x-3y)(2x+3y)(2x-2y)(3x-2y)$.

(2) 1st. $=(1+x)(1+\sqrt{x})$; 2nd. $=(1+\sqrt{x})(2x+3\sqrt{x^3})$,

$\therefore$ G.C.M. is $1+x^{\frac{1}{2}}$.

4. (1) $\frac{1}{3}\sqrt{3}-\frac{1}{6}\sqrt{6}$. (2) Extract the square root and the remainder is $17x^2y^2-cx^2y^2$, which must $=0$, $\therefore\ c=17$.

5. Left-hand member reduces to 4, $\therefore\ m=4$; $m=$ some quantity involving x would make the given expression an equation.

6. $$\frac{\sqrt{(2+x)}-\sqrt{(1+x)}}{\sqrt{(1+x)}-\sqrt{(x)}}\times\frac{\sqrt{(2+x)}+\sqrt{(1+x)}}{\sqrt{(1+x)}+\sqrt{(x)}}$$
$$=\frac{2+x-1-x}{1+x-x}=1.$$

7. (1) Reduces to $1:(x+2)-1\div(x-2)+1\div(x-1)$, whence $x=0$, or 4.

(2) Clearing second of fractions and combining result with first, $\therefore\ 7(73y-5x)=-99x+1023y$, or $8y=z$, whence $x=8$, $y=1$.

8. $x=$ distance by carriage, $y=$ distance by train;
then $x \div 10+y \div 36+(22\frac{1}{2}-x-y) \div 4=1\frac{5}{6}$; and
$x \div 4+y \div 36+(22\frac{1}{2}-x-y) \div 10=301 \div 120$;
whence $x=7\frac{1}{2}$, $y=12$.

9. See Paper II., prob. 7.

10. (1) Divide by $x-3$, remainder is $6y^2+9y+18$, which must equal 0; $\therefore\ y=\frac{1}{4}(-3\pm\sqrt{-39}$.

(2) $(a+b)=-1$, $ab=1$, $\therefore\ a^2+ab+b^2=(a+b)^2-ab=0$,
$\therefore\ a^3-b^3=(a-b)(a^2+ab+b^2)=(a-b)\times 0=0$.

Paper V., page 212.

1. $8x^3+\dfrac{1}{125}$. 2. Substitute $-y$ for x and the expression vanishes; or expression is divisible by *difference* of quantities, *i.e.*, by $(\frac{1}{2}x-y)-(x-\frac{1}{2}y)=\frac{1}{2}(x+y)$.

3. (1) Expression $\{(a+b)^2-(a-b)^2\}^2=16a^2b^2$;

(2) Expression $=(a+b+c)(a+b-c)\{c-(a-b)\}\{c+(a-b)\}$
$=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}$, expand and substitute a^2+b^2 for c^2.

4. $$\frac{(a+b)^2}{2ab}\times\frac{ab^2}{a^3+b^3}\times\frac{a^2-ab+b^2}{4a(a+b)}=\frac{b}{8a}.$$

5. Let $x=$ rate of "Rothesay" per hour in miles,
" $y=$ " "City Toronto" " "

$$\therefore\ \frac{35}{x}+\frac{35}{y}=5\tfrac{1}{4} \text{ and } \frac{42}{x-1}+\frac{42}{y}=6\tfrac{1}{2}.$$

From these equations $x=15$ and $y=12$.

6. (1) Multiply second equation by 2 and subtract,

$$\therefore x=\frac{a}{3} \text{ and } y=\frac{a}{4}.$$

(2) Add 28 to each side and transpose.

$\therefore\ x^2+5x+28-5\sqrt{(x^2+5x+28)}=24$

Put $x^2+5x+28=a^2$, $\therefore\ a^2-5a=24$, $\therefore\ a=8$ or -3, substitute in given equation, and $x=4$ or -9.

7. Let $x-1$, x and $x+1$ be the numbers,

$\therefore\ x(x-1)(x+1)=48x$, $\because$ Nos. are 6, 7 and 8, or $-6, -7, -8$,

8. (1) $m+n=-\dfrac{b}{a}$ and $mn=\dfrac{c}{a}$,

$\therefore\ b=-a(m+n)$ and $c=amn$;

Substitute these values of b and c in ax^2+bx+c and we have $ax^2+bx+c=ax^2+x\{-a(m+n)\}+amn, =a(x-m)(x-n)$.

(2) Let $r=$ each root, $\therefore\ 2r=-\dfrac{b}{a}$, $\therefore\ r=-\dfrac{b}{2a}$, which is the value of x in the second equation.

9. $\dfrac{m^2}{x^2}=\dfrac{n^2}{y^2}\ \therefore\ \dfrac{x^2}{a^2}\times\dfrac{m^2}{x^2}+\dfrac{y^2}{b^2}\times\dfrac{n^2}{y^2}=\dfrac{n^2}{y^2}=\dfrac{m^2}{x^2}=\dfrac{m^2+n^2}{x^2+y^2}$, or

$\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}=\dfrac{m^2+n^2}{x^2+y^2}$.

Paper VI., page 213.

1. Numerator of first fraction is $\{a(x^2-y^2)+2bxy\}^2$ $=a^2(x^2-y^2)^2+4abxy(x^2-y^2)+4b^2x^2y^2$: numerator of second (by symmetry) $=b^2(y^2-x^2)^2+4bayx(y^2-x^2)+4a^2y^2x^2$, $\therefore$ their sum is $(a^2+b^2)\{(x^2-y^2)^2+4x^2y^2\}=(a^2+b^2)(x^2+y^2)^2$, of which the latter factor is cancelled by the denominator.

2. (1) By common division or by factoring,

$a^2+b^2+c^2+ab+ac-bc$.

(2) This is the same form as (1), and has for *one* factor, $\{1+x+x^2)-(1-x+x^2)-2x\}=0$. Or by formula [6] the first three terms are seen to be the cube of

$(1+x+x^2)-(1-x+x^2)=(2x)^3$, &c.

3. (1) $x^2+y^2\pm\frac{3}{2}xy$. (2) $(7x+6y-9)(x-y+4)$. See Arts. XX. and XVII.

4. (1) -20. (2) 0. See Ex. 1, page 44.

5. $\sqrt{(a+x)}=\sqrt{\{a+2ab\div(1+b^2)\}}=\sqrt{a(1+b)^2}=$
$\pm(1-b)\sqrt{a}$. So second term of numerator is found to be
$\pm(1-b)\sqrt{a}$, $\therefore$ $\sqrt{a}$ cancels from both terms of resulting fraction, and result is b, or $1\div b$.

6. (1) $1\div a+a\div(1+a)=2-1\div(1+a)>1$.

(2) $(a^2+b^2)\div ab>2$, *i.e.*, if $a^2+b^2>2ab$, or $(a-b)^2>0$, which is the case, since the *square* is positive.

7. (1) From second equation, $x^{\frac{1}{2}}+y^{\frac{1}{2}}=\frac{5}{6}x^{\frac{1}{2}}y^{\frac{1}{2}}$, $\therefore$ from the first equation, $x^{\frac{1}{2}}y^{\frac{1}{2}}=6$; whence from first equation $x=4$ or 9, and from symmetry, $y=9$ or 4.

(2) Adding $(x+y-z)=6$, and $(1)-(2)$ gives $-2x+y+z=3$, thence $x=1$, $y=2$, $z=3$.

(3) Take together the first and fourth factors, and also the second and third, $\therefore$ $(x^2+7x+6)(x^2+7x+12)$,
or $(x^2+7x+6)^2+6(x^2+7x+6)=16$, $\therefore$ $(x^2+7x+6)=$
$-3\pm5=2$, or -8; $\therefore$ $x=\frac{1}{2}(-7\pm\sqrt{33})$, or $\frac{1}{2}(-7\pm\sqrt{-7})$.

8. Let $x-1$, x, $x+1$ be the numbers, then $(x-1)^3+x^3+(x+1)^3=16\frac{2}{7}\,x(x+1)$, x is seen to be a divisor of first and third cubes and $\therefore$ of the equation, $\therefore$ $x=0$; result is $7x^2-38x=24$, which gives $x=6$, or $-\frac{4}{7}$, $\therefore$ the required numbers are -1, 0, 1; 5, 6, 7; or $-\frac{11}{7}$, $-\frac{4}{7}$, $\frac{3}{7}$.

9. (1) Irrational and impossible roots enter in pairs in equations whose coefficients are real and rational, $\therefore$ the required equation is $x(x-\sqrt{-3})(x+\sqrt{-3})(x-1+\sqrt{2})\times(x-1-\sqrt{2})\times f(x)=0$, where $f(x)=0$ contains the other roots.

(2) $\sqrt{(pq)}$ is a root substituting this value for x in the equation we have $pq+p\sqrt{(pq)}+q=0$, or $q^2(p+1)^2=p^3q$, or $q(p+1)^2=p^3$.

10. Let x = number of miles per hour of train from A, and y = number of miles of train from B. Then the whole distance $=1\frac{1}{2}(x+y)$ and $1\frac{1}{2}(x+y)\div x$ = first train's time = second train's time $+52\frac{1}{2}$ minutes $=1\frac{1}{2}(x+y)\div y+\frac{7}{8}$; or by division, $1\frac{1}{2}+1\frac{1}{2}\left(\frac{y}{x}\right)=1\frac{1}{2}+1\frac{1}{2}\left(\frac{x}{y}\right)+\frac{7}{8}$, or $12y\div x=12x\div y+7$, multiply by $x\div y$, $\therefore\ \left\{\frac{x}{y}\right\}^2-\frac{7}{12}\left\{\frac{x}{y}\right\}=1$, $\therefore\ \frac{x}{y}=\frac{3}{4}$.

Paper VII., page 214.

2. First two terms $=0$ because c is a factor of each,
$\therefore$ numerical value $=-\cdot01$.

3. The quantity vanishes, when $x=-a$. or $-b$, &c. See Algebra page 41.

4. (1) Second quantity $=x^3+1+2x^2-2=(x+1)(x^2+x-1)$; the first quantity does not vanish for $x+1=0$; and x^2+x-1 is found to be the H.C.F.

(2) First expression when factored $=(x+y)\{(x-y)(ax+by)\}$;
Second " " " $=(x-y)\{(x+y)(ax-by)\}$.
$\therefore\ x^2-y^2=$ H. C. F.

5. (1) First two terms reduces to $\dfrac{12x-60}{12(4x^2-9)}=\dfrac{x-5}{4x^2-9}$,

$$\therefore \text{ whole expression } = \frac{x+5}{4x^2-9}-\frac{x-4}{4x^2+9}=\frac{36x^2+18x+9}{16x^4-81};$$

(2) Bring first two into one fraction and do similarly with second two, after cancelling we have

$$\frac{\{(x-a)(x+b(+(x+a)(x-b)\}}{\{(x-a)(x-b)+(x+a)(x+b)\}}=(x^2-ab)\div(x^2+ab).$$

6. (1) $x=9$. (2) Equation reduces to

$$(x+1)+\frac{1}{5x-4}-\left\{(x+1)+\frac{1}{7x-10}\right\}=(2x-6)\div$$

$(5x-4)(7x-10)$ $\therefore$ $x=3$.

(3) Subtract second equation from first,

$\therefore$ $x^2-y^2=ax+by-bx-ay$, $\therefore$ $(x+y)(x-y)=a(x-y)+b(y-x)$.

Divide through by $x-y$, $\therefore$ $x+y=a-b$, and since $x-y$ is a common factor, $\therefore$ $x=y$, $\therefore$ &c. It is seen from symmetry that $x=y$.

PAPER VIII., page 215.

2. (1) Product $=(x^3-y^3)^2(x-y)=x^7-x^6y-2x^4y^3+2x^3y^4+xy^6-y^7$. (2) The quantity must vanish for $x+p=0$, and for $x+q=0$, and $\therefore$ $=x^2+(p+q)x+pq$, $\therefore$ $a=p+q$, $b=pq$.

3. (1) Dividend $=(x^3+y^3)^2=(x+y)^2(x^2-xy+y^2)^2$ which divided by $(x+y)^2$, gives $(x^2-xy+y^2)^2$.

(2) Dividend $=(a^4+b^4+a^2b^2)(a^4+b^4-a^2b^2$; divisor $=$ $a^4+b^4+a^2b^2$, $\therefore$ quotient $=$ &c.

4. Let n be the number, then $n^2+\{\frac{1}{2}(n^2-1)\}^2=n^3+\frac{1}{4}(n^2=1)^2$ $=\frac{1}{4}(4n^2+n^4-2n^2+1)=\frac{1}{4}(n^4+2n^2+1)=\{\frac{1}{2}(n^2+1)\}^2$.

5. (1) Arrange in descending powers of x; *or*, by inspection.

$ax^3+bx+cx^{-1}$ (2) $\frac{1}{2}x^{\frac{3}{4}}-\frac{1}{3}x^{\frac{5}{8}}+\frac{1}{4}x^{\frac{7}{12}}$.

6. (1) Let $x+m$ be the C. M. then $m^2-am+b=0$, and $m^2-a'm-b=0$, $\therefore$ $2m^2-(a+a')m=0$, or $2m-(a+a')=0$,

$\therefore$ $m=\frac{1}{2}(a+a')$; *or* the C. M. must measure the sum of the quantites which is $2x^2+(a+a')x=x(2x+a+a_,)$.

Also, $m^2-am+b=0$, $m^2-a'm-b=0$; express b in terms of a, a'; subtract equations, $\therefore$ $(a-a')m-2b=0$, or $2b=(a-a')m$ $=(a-a')\times\frac{1}{2}(a+a')$, $\therefore$ $4b=a^2-a'^2$.

(2) Factors of first are $x^2+p^2\pm px$; the second $=(x^2+px)^2-p^4$ $=(x^2+px+p^2)(x^2+px-p^2)$, $\therefore$ x^2+px+p^2 is H. C. F.

(3) first $= \frac{5}{2}(x+5)(x-4)$, second $= \frac{10}{3}(x+5)(x-6)$, third $= \frac{25}{6}(x-4)(x-6)$, $\therefore$ L.C.M. $= 50(x+5)(x-4)(x-6)$.

7. $a=0$, $b=0$; or $a=1$, $b=2$. Let $m=$ value of fraction for all values of x, $\therefore\ 3x^2-(4a+b)x+a+2b^2=$ $m\{5x^2-(8a+b)x-a+4b^2\}$, in which coefficients of like powers of x are equal, $\therefore\ 5m=3$, $4a+b=(8a+b)m$, $a+2b^2=(-a+4b^2)m$, whence, &c.

8. Transpose and square, the equation reduces to

$16(x+1)(x+6)=9x^2+90x+225$, and $x=3$, or $-43\div 7$.

Apparenily neither value satisfies the equation; but any quantity has *two* square roots; and if the negative root of $\sqrt{(x+1)(x+6)}$ be taken, the value 3 satisfies the equation.

PAPER IX., page 217.

1. Quantity $=(a^2+n^2)(2an+a-2an-n)+$
$an(2n+1-2a-1)=(a^2+n^2)(a-n)-2an(a-n)=$
$(a-n)(a^2+n^2-2an)=(a-n)^3$.

3. Let quotients be $x+r$ and $x+s$, respectively, so that
$x^3+ax^2+b=(x^2+mx+n)(x+r)=$
$x^3+(m+r)x^2+(n+mr)x+nr$, and $x^3+px+q=$
$(x^2+mx+n)(x+s)=x^3+(m+s)x^2+(n+ms)x+ns$.

Equating coefficients,

(1) $a=m+r$.	(4) $m+s=0$,
(2) $n+mr=0$.	(5) $n+ms=p$.
(3) $b=nr$.	(6) $ns=q$.

(3)$-$(6), $b-q=n(r-s)=na$, by (1)$-$(4);

$\therefore\ (b-q)^3=n^3a^3$. Also, (1)$\times$(6), $bq=n^2rs=n^3$ [from (2) and (4)], $\therefore\ (b-q)^3=a^3bq$.

4. First quantity $=(axy+b)(ax^{-1}y^{-1}-b)$; the second $=$ $(axy+b)(ax^2y^{-1}+abx-bxy^{-2}-b^2y^{-1})$, $\therefore\ axy+b$ is H.C.F.

5. Let a = left hand digit of A, and b the right; let c = left-hand digit of B, and d the right; $\therefore$ $10a+b=A$, and $10c+d=B$;

$a-c=x$, $10a+b-10c-d=y$, $c+d-a-b=z$;

$\therefore$ $9a-9c=y+z=9x$. A, 77; B, 69.

6. Let each ratio $=x$; $\therefore$ $a=bx$, $b=cx$, $c=dx$; $\therefore$ $abc=bcdx^3$, and $x=\sqrt[3]{\frac{a}{d}}$. Also $a+b+c=(b+c+d)x$, $\therefore$ &c.

7. (1) Equation, on division, &c., becomes

$$\frac{a}{x-a}+\frac{a}{x+a}=\frac{x}{b-x}+\frac{x}{b+x}, \text{ \&c.} \quad x=\pm\sqrt{ab}.$$

(2) $y=\sqrt{(a-b)}\div\sqrt{(a+b)}$, $x=\sqrt{(a+b)}\div\sqrt{(a-b)}$.

8. $P\div a$ = cost of each; $\frac{21}{20}.bP\div a$ = selling price of b sheep, $\therefore$ $\left(\frac{11P}{10}-\frac{21bP}{20a}\right)\div(a-b)$ = selling price of each remaining sheep, $=P(22a-21b)\div 20a(a-b)$.

9. x = 1st, y = 2nd, $\therefore$ $70-(x+y)$ = 3rd; then $y\div x=2+1\div x$,

or $y=2x+1$; $(70-x-y)\div y=3+3\div y$,

$\therefore$ $x=y=15$; $\therefore$ 7, 15, 48.

Paper X., page 218.

1. Dividend is $(ax^2+by^2+2cxy)(x+y+z)$, $\therefore$ &c.

2. Quantity must vanish when $x=-1$, $\therefore$ $1+p-q+a^2=0$; also when $x=1$, $\therefore$ $1+p+q+a^2=0$, from these two $q=0$ and $1+p+a^2=0$. Second, the quantity must vanish when $x=a$, $\therefore$ $a^4+pa^2+qa+a^2=0$ and also when $x=-a$, $\therefore a^4+pa^2-qa+a^2=0$, and these two give $q=0$, and $1+p+a^2=0$, the same as before. *Or* substitute in given expression the values of p, q, from former, and result is plainiy divisible by x^2-a^2.

3. 3(first quantity) − second $=3x^4-10x^2-8=(3x^2+2)(x^2-4)$, of which first factor is not a C.M. $x^2-4=(x+2)(x-2)$; the quantities vanish when $x-2=0$, but not for $x+2=0$, $\therefore$ $x-2$ is H. C. F.

4. Let x = distance, then $\frac{x}{12}+\frac{2x}{9}+\frac{2}{3}=\frac{x}{6}+\frac{x}{8}+1$, $x=24$.

5. (1) Square root of first term $=5x^2$, of last $4a^2$, *twice* product is $40x^2a^2$; but expression contains $49x^2a^2$, $\therefore 9x^2a^2$ is square of third term, $\therefore 5x^2-3ax+4a^2$. See example 4, page 62 of Hand-Book.

(2) Expression $= \frac{x^2}{y^2}+2+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt{2}+\frac{1}{2}$

$=\left(\frac{x}{y}+\frac{y}{x}\right)^2-\ldots\ldots=\left(\frac{x}{y}+\frac{y}{x}-\sqrt{\frac{1}{2}}\right)^2$, $\therefore$ square root $=$ &c.

7. (1) $x+1+\frac{1}{5x-4}=x+1+\frac{1}{7x-10}$, $\therefore 7x-10=5x-4$; $x=3$

(2) $(4x-2)^2=(5x-3)^2-(3x-1)^2=2(2x-2)(4x-2)$, $\therefore$ $4x-2=0$, $x=\frac{1}{2}$.

8 Let $2x$, $3x$, be number sides respectively, then $4x-4=$ number right angles in first. $6x-4=$ number in second; $\therefore$ $(4x-4)\div 2x$, and $(6x-4)\div 3x$ are the magnitudes of one angle in each, taking one right angle as unit, $\therefore (4x-4)\div 2x : (6x-4)\div 3x$, :: $3:4$, and $2x=4$, $3x=6$.

PAPER XI., page 219.

1–2. 2. Let x = time for A, y = time for B; then $1\div x$ = part done by A in 1 day, and $1\div y$ = part done by B. Also $4\div 3x$ and $3\div 2y$ represent parts per day on second supposition,

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{7}{24}, \text{ and } \frac{4}{3x}+\frac{3}{2y}=\frac{59}{144}, \text{ or}$$

$$\frac{1}{2y}=\frac{1}{16}, \therefore y=8, \quad x=6.$$

3. (1) 3rd factor $=(2x^{\frac{1}{4}}+3y^{\frac{1}{4}})^2$; 4th $=(2x^{\frac{1}{4}}-3y^{\frac{1}{4}})^2$; $\therefore$ entire product $=(2x^{\frac{1}{4}}+3y^{\frac{1}{4}})^3\times(2x^{\frac{1}{4}}-3y^{\frac{1}{4}})^3=(4x^{\frac{1}{2}}-9y^{\frac{1}{2}})^3=$ &c.

(2) $=(\frac{1}{4}x^2+\frac{2}{9}y^2+\frac{1}{3}xy)(\frac{1}{4}x^2+\frac{2}{9}y^2-\frac{1}{3}xy)=\frac{1}{16}x^4+\frac{4}{81}y^4.$

4. (1) x^2+3+9x^{-2}. (2) By Horner's Division, or by factoring, $x^2-(a+b)x-c$.

5. $=x^{2n-1}(x^{2m-2n+2}-1)$, which vanishes for $x=1$, and $x=-1$, since $2m-2n+2=2(m-n+1)$ is necessarily *even*.

6. Add the fractions; numerator =

$$(a^2-bc)(b+c)+(b^2-ca)(c+a)+(c^2-ab)(a+b),$$

which vanishes for $a=0$, and $\therefore$ for $b=0$, and $c=0$. Also it vanishes for $a+b=0$, $\therefore$ for $b+c=0$, and $c+a=0$, *i.e.*, it vanishes for *six* differe t factors, $\therefore$ it vanishes identically.

7. (1) Equation reduces to $\frac{1}{x+1}+\frac{4}{x+4}=\frac{2}{x+2}+\frac{3}{x+3}$, or $-x\div(x+1)(x+2)=-x\div(x+3)(x+4)$, $\therefore$ $x=0$, or $-2\frac{1}{2}$.

(2) $\frac{21}{4}\left\{(x^2+y^2)\frac{x}{y}\right\}=\frac{100}{3}\left\{(x^2-y^2)\frac{y}{x}\right\}$ or $63x^4+63x^2y^2=400x^2y^2-400y^2$, or $(7x^2-25y^2)(9x^2-16y^2)=0$, $\therefore$ $9x^2-16y^2=0$, and $x=\pm\frac{4}{3}y$, substitute in first equation, $y=\pm 3$, or $\pm\sqrt{-9}$

PAPER XII., page 220.

2. $(a+b)(a-b)=a^2-b^2$; $(a-b)(a+b)(a^2+b^2)=a^4-b^4$, &c., *i.e.*, the product of *three* factors $=a^4b^4$; of *four* $=a^8-b^8$ of *five*, $=a^{16}-b^{16}$ .... ; $\therefore$ of $n+1$, a^n-b^n.

3. Apply Horner's method,

$$\begin{array}{c|cccccc}
2 & 1 & -1 & & & & \\
 & & +2 & +2 & +4 & +8 & \ldots\ldots \\
\hline
 & 1 & +1 & +2 & +4 & +8 & +\ldots\ldots
\end{array}$$

or $1+x+2x^2+4x^3+8x^4+$............... In this series note that

(1) the index of x in any term is *one less* than the number of the term; (2) the coefficient of any term is that power of 2 whose index is *two less* than the number of the term.

(3) the remainder after any term = next following term.

$\therefore$ $(r+1)$th term $=2^{r-1}x^r$, and the required remainder $=2^rx^{r+1}$.

4. x and $3-x=$ the parts, then $x^2-(3-x)^2=(x+3-x)\times\{x-(3-x)\}=3\{x-(3-x)\}$.

5. (Difference of the quantities) $\div 6x=1-5x+6x^2$. which divides the second quantity, and also the first giving quotient, $1-3x-4x^2$, $\therefore$ L.C.M. $=(1-3x-4x^2)(1-2x-13x^2+38x^3-24x^4)$.

6. Expression is $x^4+2mx^3+x^2(m+2n+p)+x(2mn+q)+n^2$; the first two terms of the root must be x^2+mx and the last term n, $\therefore$ the expression must be $(x^2+2mx+n)^2=x^4+2mx^3+x^2(m^2+2n)+2mnx+n^2$, equating coefficients we have $m^2+2n=m+2n+p$, $\therefore$ $m^2-m=p$, and $2mn=2mn+q$, $\therefore$ $q=0$; and n is independent of the others and may have any relation to them.

7. (1) Reduce the fractions and add, $\therefore$ $2+4x^2=a(1-x^2)$, $x^2=(a-2)\div(a+4)$. (2) Reduce first fraction, $\therefore$ $\sqrt{ax}-b=(\sqrt{ax}-b)\div n-c$, $\therefore$ $\sqrt{x}=\{b(n-1)-cn\}\div\sqrt{a}(n-1)$.

(3) Equations reduce to $\frac{1}{x}+\frac{1}{y}=\frac{2}{z}+\frac{2}{x}=\frac{3}{z}+\frac{3}{y}=1$, $\therefore$

$3\left(\frac{1}{y}-\frac{1}{z}\right)=\frac{3}{2}$ and $3\left(\frac{1}{y}+\frac{1}{z}\right)=1$, $\therefore y=2\frac{2}{5}$; $x=1\frac{5}{7}$, $z=-12$.

8. $\frac{x+y+z}{z}=\frac{ax}{z}=\frac{by}{z}=\frac{a(y+z)}{(a-1)z}=\frac{abz}{\{(a-1)b-a\}z}=\frac{ab}{(a-1)(b-1)-1}$.

9. $x=$tens, y the units, then $10x+y=$ the number, $\therefore$ $x+y=10$, $20x+2y-1=10y+x$, whence $x=$, $y=7$.

Paper XIII., page 221.

1. The factors of the numerator are a^2-b^2, b^2-c^2 c^2-a^2, $\therefore$ the expression $=$

$$\{(a^2-b^2)(b^2-c^2)(c^2-a^2)\}\div\{(a+b)(b+c)(c+a)\}$$
$$=(a-b)(b-c)(c-a)=(2\tfrac{1}{2}-3\tfrac{1}{2})(3\tfrac{1}{2}-4\tfrac{1}{2})(4\tfrac{1}{2}-2\tfrac{1}{2})=2.$$

2. $a-b$ becomes $\{(a+x)-(b+x)\}=a-b$, $\therefore$ $b-c$ and $c-a$ also remain the same as before. $\therefore$ the expression would still be $(a-b)(b-c)(c-a)$, which $=2$, when $a=2\frac{1}{2}$, b, $3\frac{1}{2}$; c, $4\frac{1}{2}$.

3. $(1+a_1x)(1+a_2x)\ldots\ldots\ldots\ldots(1+a_nx)$. The first term of product is evidently 1; to obtain the second term a_1x would be multiplied into the product of the first terms of all the factors, leaving out the first; this gives a_1x; similarly, a_2x multiplied into the product of the first terms of all the factors excepting the second $=a_2x$, &c.

$\therefore$ the second term $=(a_1+a_2+\ldots\ldots\ldots+a_n)x$, the third $=$ $(a_1a_2+a_1a_3+\ldots\ldots\ldots+a_1a_n+a_2a_3+a_2a_4+\ldots\ldots\ldots)x^2$

$\therefore$ $(1+a_1x)(1+a_2x)\ldots\ldots\ldots(1+a_nx)=1+C_1x+C_2x^2+\ldots\ldots$ where C_1 is the sum of the combinations of $a_1\ a_2\ a_3\ldots\ldots a_n$ taken one at a time; C_2 the sum of combinations of $a_1\ a_2\ a^3\ldots\ldots a_n$ taken two at a time.

4. Let $x=$ sum lent at $8\frac{1}{2}$ per cent.,

$y=$ " 9 "

then $x+y=$ whole sum borrowed;

$$\frac{8\frac{1}{2}x}{100}+\frac{9y}{100}=\frac{y}{100}(x+y)+75$$

$$\frac{8\frac{1}{2}y}{100}+\frac{9x}{100}=\frac{y}{100}(x+y)+65$$

$$\frac{8\frac{1}{2}}{100}(x+y)+\frac{9}{100}(x+y)=\frac{14}{100}(x+y)+140$$

$$\frac{3\frac{1}{2}}{100}(x+y)=140,\quad x+y=\$4000.$$

5. $ax+by=c$, $a'x+b'y=c'$ $x=\dfrac{b'c-bc'}{ab'-a'b}$, $y=\dfrac{ca'-c'a}{a'b-ab'}$,

$$\therefore\ mx+ny=\frac{m(b'c-bc')-n(ca'-c'a)}{ab'-a'b}.$$

This value is indeterminate when $m(b'c-bc')=n(ca'-c'a)$ and $ab'=a'b$, *i. e.*, when $\dfrac{m}{n}=\dfrac{ca'-c'a}{b'c-bc'}$ and $\dfrac{a}{a'}=\dfrac{b}{b'}$.

6. $\dfrac{a_1}{a_2}=\dfrac{a_1+a_2+\ldots\ldots\ldots+a_{n-1}}{a_2+a_3+\ldots\ldots\ldots+a_n}$.

$\therefore\ a_1(a_2+a_3+\ldots\ldots+a_n)-a_2(a_1+a_2+\ldots\ldots a_{n-1})=0,$

$\therefore\ a_1(a_1+a_2+\ldots.+a_n)-a_2(a_1+a_2+\ldots..+a_{n-1}+a_n)=a_1^2-a_2a_n,$

$\therefore\ (a_1-a_2)(a_1+a_2+\ldots\ldots+a_n)=a_1^2-a_2a_n,$

$\therefore\ a_1+a_2+a_3+\ldots\ldots\ldots\ldots a_n=\dfrac{a_1^2-a_2a_n}{a_1-a_2}$.

7. $\alpha+\beta=(x^{\frac{1}{3}}+y^{\frac{1}{3}})^3\ \therefore\ \sqrt[3]{(\alpha+\beta)}=x^{\frac{1}{3}}+y^{\frac{1}{3}},$

$\alpha-\beta=(x^{\frac{1}{3}}-y^{\frac{1}{3}})^3\ \therefore\ \sqrt[3]{(\alpha-\beta)}=x^{\frac{1}{3}}-y^{\frac{1}{3}},$

$\therefore\ x^{\frac{1}{3}}=\frac{1}{2}\{(\alpha+\beta)^{\frac{1}{3}}+(\alpha-\beta)^{\frac{1}{3}}\}\quad x^{\frac{2}{3}}=\frac{1}{4}\{(\alpha+\beta)^{\frac{1}{3}}+(\alpha-\beta)^{\frac{1}{3}}\}^2$

$y^{\frac{2}{3}}=\frac{1}{4}\{(\alpha+\beta)^{\frac{1}{3}}-(\alpha-\beta)^{\frac{1}{3}}\}^2\quad\therefore\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=\frac{1}{2}\{(\alpha+\beta)^{\frac{2}{3}}+(\alpha-\beta)^{\frac{2}{3}}\}$

$\therefore\ (\alpha+\beta)^{\frac{2}{3}}+(\alpha-\beta)^{\frac{2}{3}}-2a^{\frac{2}{3}}=0.$

8. $aa_1x^2+ba_1x+ca_1=0\quad(3)\qquad aa_1x^2+ab_1x+ac_1=0\quad(4),$

$(3)-(4)\ \therefore\ (a_1b-ab_1)x=ac_1-a_1c,\ \ \therefore\ x=\dfrac{ac_1-a_1c}{a_1b-ab_1},$

Again, $ac_1x^2+bc_1x+cc_1=0\quad(5)\qquad a_1cx^2+b_1cx+cc_1=0\quad(6)$

$(5)-(6)\ (ac_1-a_1c)x^2=(b_1c-bc_1)x$ or $(ac_1-a_1c)x=(b_1c-bc_1)$

$x=\dfrac{b_1c-bc_1}{ac_1-a_1c}\ \therefore\ \dfrac{b_1c-bc_1}{ac_1-a_1c}=\dfrac{ac_1-a_1c}{a_1b-ab_1},$

$\therefore\ (ab_1-a_1b)(bc_1-b_1c)=(ac_1-a_1c)^2.$

9. Let $x=$ left-hand digit, $y=$ right-hand digit,

$\therefore\ (10x+y)+(10y+x)=121,\quad\therefore\ x+y=11\qquad(3)$

$(10x+y)-(10y+x)=9,\qquad\therefore\ x-y=1\qquad(4)$

From (3) and (4) $x=6$, $y=5$, $\therefore$ number required is 65.

PAPER XIV., page 222.

1. The expression vanishes when $a=0$, $\therefore$ a is a factor, and by symmetry, b and c are factors; $\therefore$ the expression $=nabc$ where n is numerical.

Let $a=b=c=1$, and then $n=12$, $\therefore$ the expression $=12abc$.

3. $x^2+y^2+z^2=-2(xy+yz++zx)$

$-2(x'y'+y'z'+z'x')=x'^2+y'^2+z'^2$,

$\therefore (x^2+y^2+z^2)(x'y'+y'z'+z'x')=(x'^2+y'^2+z'^2)(xy+yz+zx)$,

hence $x'^2yz-x^2y'z'+y'^2zx-y^2z'x'+z'^2xy-z^2x'y'$ (1)

$=x^2x'(z'+y')-xx'^2(z+y)+y^2y'(x'+z')-yy'^2(z+x)$

$+z^2z'(y'+x')-zz'^2(y+x)$

$=x^2x'(-x')-xx'^2(-x)+y^2y'(-y')-yy'^2(-y)+z^2z'(-z')$

$-z'z^2(-z')=x^2x'^2-x^2x'^2+\&c.=0$.

$a^2(yz-y'z')+b^2(zx-z'x')+c^2(xy-x'y')$

$=(x^2+x'^2)(yz-y'z')+(y^2+y'^2)(zx-z'x')$

$+(z^2+z'^2)(xy-x'y')$

$=xyz(x+y+z)-x'y'z'(x'+y'+z')$

$+(x'^2yz-x^2y'z'+y'^2zx-y^2z'x'+z'^2xy-z^2x'y')=0$.

4. Let $x+$dist. and $y=$ Q's rate, then $\frac{1}{2}x\div 4=\frac{1}{2}x\div(y-1)$

$\therefore$ $y=5$, also $\frac{1}{2}x\div 3=\frac{1}{2}x\div y+\frac{32}{60}$, $\therefore$ $\frac{1}{6}x=\frac{1}{10}x+\frac{32}{60}$, $x=8$ mls.

5. (1) $x(mn-n^2)=am^2-amn$, $x=\dfrac{am}{n}$.

(2) Add twice the second equation to the first, then $(x^2+y^2)^2+(x^2+y^2)-182=0$, $\therefore$ $(x^2+y^2-13)(x^2+y^2+14)=0$,

$\therefore$ $x^2+y^2=13$ or $x^2+y^2=-14$.......................... (3)

$\therefore$ $x^2y^2=49-13$ or $49+14$, $xy=\pm 6$ or $\pm 3\sqrt{7}$ ...... (4)

taking $x^2+y^2=13$ and $xy=\pm 6$, $(x+y)^2=25$ or 1,

$\therefore x+y=\pm 5$ or ± 1, $(x-y)^2=1$ or 55, $x-y=\pm 1$ or ± 5, &c.

6. Let x and y be the numbers, then $\frac{1}{2}(x-y)=\frac{1}{3}(x+y)=\frac{2}{5}x$. $\frac{1}{2}(x-y)=\frac{1}{3}xy=\frac{2}{5}x$, $\therefore\ \frac{2}{5}x=\frac{1}{5}xy$, $\therefore$ since x cannot $=0$, $y=2$,

$$\frac{x+y}{3}=\frac{x-y}{2}=\frac{2y}{1}=4=\frac{2x}{5},\ \therefore x=10.$$

Paper XV., page 222.

2. From formula [8] the expression $=0+(c-a)^3=\frac{1}{8}$.

3. $c^2(c^2-bd)+d^2(b^2-ac)+ac(ad-bc)$.

4. Let x represent the time after noon, then $mx=nx+2$, $\therefore x=2\div(m-n)$. If $m-n$ is negative, x is negative, which shows that they were together *before* noon. If $m-n=0$, x is infinite, which shows that they are never together.

5. (1) $(x+a)(x-2a)$, $x^2(x+a)$, $-x^2(x-a)$, $\therefore$ the L C.M. is $x^2(x+a)(x-a)(x-2a)$.

(2) Quantities are $(x-a)(x+a)(x-y)$, and $(x+a)(x+y)(x-y)$, $\therefore$ L. C. M. is $(x^2-a^2)(x^2-y^2)$.

6. (1) First fraction reduces to $(a+b-c+d)\div(a+b+c+d)$, $\therefore$ by symmetry the denominators of the others are the same, and the numerators also can be written down at once.

$\therefore$ result is $(a+b+c+3d)\div(a+b+c+d)$.

(2) $(x+y+z)\div(x-y+z)$; $3a\div(a+b)$.

7. Multiply sum of first and second by b, and arrange result :

$\therefore\ ab(x+y-z)+bc(x-y-z)=2b^2(c-a)+ab(a+b)-bc(b+c)$. and by symmetry two similar results may be written down, add the three results, $\therefore\ 2a^2(b-c)+2b^2(c-a)+2c^2(a-b)=0$.

8. $10x+y=P$, $10y+x=Q$, $\therefore\ P+Q=11(x+y)$, and $P-Q=9(x-y)$, $\therefore\ 11(x+y)\times 9(x-y)=9(x-y)\times 11(x+y)$, an identity.

PAPER XVI., page 224.

1. $(a^2+b^2)(x^2+y^2)(a^2-b^2)(x^2-y^2)=(a^4-b^4)(x^4-y^4)$;

$\{(a-b)+(b-c)+(c-a)\}^2=0,$

$\therefore\ (a-b)^2+(b-c)^2+(c-a^2)+2(a-b)(b-c)+2(a-b)(c-a)+$
$2(b-c)(c-a)=0,$

$\therefore\ (a-b)^2+(b-c)^2+(c-a)^2=2(a-b)(a-c)+2(b-c)(b-a)+$
$2(c-b)(c-a).$

2. $\dfrac{s-a}{s}+\dfrac{s-b}{s}+\dfrac{s-c}{s}+$ &c. to n *terms* $=$

$$\frac{ns-a-b-c \text{ to } n \text{ terms}}{s}=\frac{ns-s}{s}=\frac{s(n-1)}{s}=n-1.$$

3. First let $a-b$, $b-c$, $c-a=$ three positive quantities $k\ l\ m$. Adding, $0=k+l+m$ sum of three positive quantities zero, which is impossible. Similarly, $a-b$, $b-c$, $c-a$, can be proved not all negative.

4. $2x^4-3x^3+4x+3$.

5. See Paper XIX., prob. 4. The value of $(p-q)^2$ found as there indicated, will vanish if any of the given conditions hold, unless *also* $a+b=c+d$, in which case the value is $0\div 0$.

6. $\therefore\ x^{2n}\div ay^{2n}=x^n\div b(x-y)^n$

$$\therefore\ \frac{x^2}{y^2}-\frac{x}{y}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \text{ and } \frac{x}{y}=\tfrac{1}{2}\left\{1\pm\sqrt{\left(\frac{a^{\frac{1}{n}}+4b^{\frac{1}{n}}}{4b^{\frac{1}{n}}}\right)}\right\}.$$

7. See Prob. 12, Ex. XLI.

8. Let $x=$ distance from the station (P.) to the point where A overtook B, and $x-c=$ distance from station (Q.)

$$\therefore\ x\div a=(x-c)\div b,\ x=ac\div(a-b).$$

If $a<b$, x is negative and must be reckoned to left of P, and A must have overtaken B before they reached ihe points P, Q, respectively, or they must have been moving in the opposite direction, and B must have overtaken A at some point to the left of P.

Paper XVII., page 225.

1. $1 - \dfrac{a+b}{ab} = \dfrac{ab-(a+b)}{ab}$.

2. (1) $=\{(+b)^2-a\}\{(x-b)^2-a\}$

$$=(x^2-b^2)^2-a\{(x+b)^2+(x-b)^2\}+a^2$$

$$=x^4-2x^2(a+b^2)+(a-b^2)^2.$$

(2) $8a^2 \quad 2ab-10ac-3b^2+2c^2+5bc$.

3. (1) If $x+a$ is a factor of x^2+px+q,

then $x^2+px+q=(x+a)\left(x+\dfrac{q}{a}\right)$,

hence $p=a+\dfrac{q}{a}$, $\therefore\ \dfrac{q}{a}=p-a$.

Hence, $x^2+px+q=(x+a)(x+p-a)$.

Similarly, $x^2+p x'+q'=(x+a)(x+p'-a)$.

L. C. M. = product divided by H. C. F.,

$$=\frac{(x+a)(x+a)(x+p-a)(x+p'-a)}{x+a}$$

$$=(x+a)(x+p-a)(x+p'-a).$$

(b) Difference $=\dfrac{x-a}{x-a}+\dfrac{x-b}{x-b}+\dfrac{x-c}{x-c}=3$, which is independent of x.

4. By Art XXXVII.,

$$\frac{bx+cy+dz}{b+c+d-a}=\frac{(a+b+c+d)x+\text{anal.}+\text{anal.}}{2(a+b+c+d)}=\frac{x+y+z}{2}.$$

5. Substituting 3 for x, the two sides of the equation become equal.

6. $x^3+\dfrac{1}{x^3}+3\left(x+\dfrac{1}{x}\right)=\left(x+\dfrac{1}{x}\right)^3$ Section II. [5].

Hence $\left(x+\dfrac{1}{x}\right)^3=m$, and $\left(x-\dfrac{1}{x}\right)^3=n$,

$$\therefore\ x+\frac{1}{x}=\sqrt[3]{m},\text{ and } x-\frac{1}{x}=\sqrt[3]{n};$$

$$2x = \sqrt[3]{m} + \sqrt[3]{n}\text{; and } \frac{2}{x} = \sqrt[3]{m} - \sqrt[3]{n}\text{; } \therefore m^{\frac{2}{3}} - n^{\frac{2}{3}} = 4.$$

7. If $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = \frac{1}{a+b-c}$,

Then $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} - \frac{1}{a+b-c} = 0.$

Now, let $a = -b$, and expression becomes zero;

Hence $a+b$ is a factor,

Similarly $a-c$ "

And $b-c$ "

Hence, expression $= m(a+b)(a-c)(b-c) = o$;

$\therefore$ one, at least, of factors $= o$;

$\therefore a, b, c$, cannot be all different.

8. Let $x =$ number of gallons drawn from first cask.

Then $b+c-x =$ " " " second cask.

Hence $\frac{m}{m+n}x + \frac{p}{p+q}(b+c-x) = b.$

$$\therefore x = \frac{bq-pc}{mq-pn}(m+u),$$

$$\text{and } b+c-n = \frac{mc-bn}{mq-pn}(p+q).$$

Paper XVIII., page 226.

1. (1) $(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8) = 1-x^{16}$.

(2) Theory of Divisor, page 39.

2. Since $a(b-c)^2 - c(b+c)^2 = o$

$$\frac{a}{c} = \left(\frac{b+c}{b-c}\right)^2 \quad \frac{\sqrt{a}}{\sqrt{c}} = \frac{b+c}{b-c} \quad \frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}} = \frac{b}{c};$$

$$\therefore \frac{c}{b} \cdot \frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}} = \frac{b}{c} \times \frac{c}{b} = 1.$$

3. Let $x =$ A's age, and $z =$ B's age.

$$\therefore x-y = \tfrac{3}{5}(x+y)\text{; } \therefore x = 4y.$$

Also, father's age $= 2y + 5x$, and mother's age $= 5x - 2y$,

$$\text{and } \frac{5x-2y}{5x+2y} = \frac{20y-2y}{20y+2y} = \tfrac{9}{11}.$$

$\therefore$ mother's age is $\frac{9}{11}$ of father's.

4. The difference between the quantities is

$(a-b)x^2 - 2(a^2-b^2)x + (a+b)(a^2-b^2) = (a-b)\{x-(a+b)\}^2$

and the H. C. F. is a factor of this difference. But it cannot be $a-b$, for x^3 is not divisible by $a-b$; neither can it be $\{x-(a+b)\}^2$, for $(a+b)^2$ will not divide $a^2(a+b)$ nor $b^2(a+b)$. Hence, if the quantities have a common measure it must be $x-a-b$, and this, by trial, is found to be a factor.

5. Identity holds if $(a+b-c)^3+(c+a-b)^3 =$
$2\{(b+c)^3-(b+c-a)^3\}$, if $(a+b-c)^2-(a+b-c)(c+a-b)$
$=(b+c)^2+(b+c)(b+c-a)+(b+c-a)^2$, if $a(b+c)=4bc$. Or add and subtract $(a+b+c)^3$, and apply identity in q. 7, page 36, Hand-Book.

6. (1) Bookwork.

(2) Multiply out and extract the root in the ordinary way; tne vault is $x^2-xy-xz-yz$.

7. $$\frac{x-y}{1+xy} = \frac{\frac{a+b}{a-b} - \frac{b}{a}}{1+\frac{a+b}{a-b}\cdot\frac{b}{a}} = \frac{a^2+b^2}{a^2+b^2} = 1.$$

8. (1) $x=9a$.

(2) Square and simplify, and

$$4x^4-8x^3+3=0$$

or $(2x^2-1)(2x^2-3)=0$

$$\therefore (2x^2-1=0, \text{ or } x=\pm\frac{1}{\sqrt{2}}$$

$$\text{and } 2x^2-3=0, \text{ or } x=\pm\sqrt{\tfrac{3}{2}}=\tfrac{1}{2}\sqrt{6}.$$

9. Let $x=$ one part, then $21-x=$ the other;
$10x=9(21-x)+1$; $\therefore x=10$ and $21-x=11$.

PAPER XIX., page 227.

1. $=\frac{1}{4}(x-2a+1)(4x-a+1)\times\frac{1}{4}(x+4a-2)(4x-a+1)\div \frac{1}{4}(x-2a+1)(x+4a-2)=\frac{1}{4}(4x-a+1)^2$.

2. From the conditions $xy+yz+zx=0\ldots\ldots(1)$, and $x^2+y^2+z^2=0\ldots\ldots(2)$.

Also by formula [8] and (2),

$$(x^2+y^2+z^2)^3=x^6+y^6+z^6-3x^2y^2z^2=0\ldots\ldots(3).$$

$(x+y+z)^3=x^3+y^3+z^3-3xyz=0\ldots(4)$ now divide (3) by (4).

3. (1) $(5x-1)(4x+1)$, and $(5x^2-1)(15x^2+3x+3)$, $\therefore\ 5x^2-1$ = H. C. F.

(2) $(x+y)^7=7xy(x+y)(x^2+xy+y^2)^2$,

$(x^3-y^3)^2=(x-y)^2(x^2+xy+y^2)^2$,

$\therefore$ the latter factor is the H. C. F.

4. First eliminate xy, and find the value of $x+y$, hence $(x+y)^2$ is known; second, eliminate $x+y$, and find the value of $4xy$, which subtract from that of $(x+y)^2$ and $(x-y)^2$ is found. Or,

$ab-(a+b)(x+y)+4xy=0\ldots\ldots(1)$,
$cd-(c+d)(x+y)+4xy=0\ldots\ldots(2)$,
and let $(x-y)^2=z\ldots\ldots(3)$.

(2) − (4) $\quad\therefore\ (a+b-c-d)(x+y)=ab-cd\ldots\ldots(4)$

(1) + (3) $\quad\therefore\ ab-(a+b)(x+y)+(x+y)^2=z$,

or $\{a-(x+y)\}\{b-(x+y)\}=4\ldots\ldots(5)$.

But $(a+b-c-d)\{a-(x+y)\}$
$=a^2-ac-ad+cd=(a-c)(a-d)\ldots\ldots(6)$.

And $(a+b-c-d)\{b-(x+y)\}=(b-c)(b-d)\ldots\ldots(7)$.

(5), (6), (7) $\quad\therefore\ (a+b-c-d)^2z=(a+b-c-d)^2\{a-(x+y)\}\times\{b-(x+y)\}=(a-c)(a-d)(b-c)(b-d)\ldots\ldots(8)$.

(3), (8) $\quad\therefore\ (x-y)^2=(a-c)(a-d)(b-c)(b-d)\div(a+b-c-d)^2$.

5, Add (1) and (2) and z is found at once $=bx+ay$, which substitute in (2), and we get

$y^2(1-a^2)=x^2(1-b^2)=$ (by symmetry) $z^2(1-c^2)$
$\therefore\ x^2\div(1-a^2)=y^2\div(1-b^2)=z^2\div(1-c^2)$.

6. Apply (5) page 122. $\quad x=(a+2)^2\div(4a^2+4a)$.

7. Extract square root, remainder is

$$-x(3a^3+3ab^2)+(2b^2)^2-\tfrac{1}{4}(3b^2-a^2)^2,$$

whence $x=(7b^2-a^2)\div 12a$.

8. Let x = number women, then $x+4$ = number men, $2x+14$ = number children $\therefore\ 4x+18=90$, $x=18$.

PAPER XX., page 228.

1. $(1+m)x+(1-n)y$. (2) The remainder is $(m^2-n-pm+q)x+mn-pn+r$, which must vanish for all values of x, $\therefore$ $m^2-n-pm+q=0$, or $p=(m^2-n+q)\div m$; also $mn-pn+r=0$, $\therefore\ p=(mn+r)\div n$; equate these two values of p, and

$$nq-n^2=rm.$$

3. $(x^2-1)(x^2-px+q)$, and $(x^2-1)(x^2-qx+p)$, $\therefore\ x^2-1$ is the H. C. F.

4. (1) $=(x+y-z)(y+z-x)(z+x-y)\div(x+y+z)^3$.

(2) $\{a^2b-2ab^2-a^3+ab^3+b^3\}\div(a^4-b^4)$.

5. Equation reduces to $2\div(8x-9)-1\div(4x-3)=$

$1\div(4x-5)-2\div(8x-7)$, whence $x=1$.

6. $a''(cb'-bc')+b''(ac'-a'c)+c''(a'b-ab')=0$.

7. (1) $x=\sqrt{\left\{n^2-\left(\dfrac{m^3-2n}{3m}\right)^3\right\}}$. (2) $x=2$, $y=3$, $z=4$.

8. Let x = number of first kind, $\therefore\ c-x$ = that of second,

$\therefore\ \dfrac{x}{a}+\dfrac{c-x}{b}=1$, whence $x=a(b-c)\div(b-a)$,

$c-x=b(c-a)\div(b-a)$.

9. Let x miles per hour = rate at starting, $\therefore\ 4\frac{1}{2}\div x$ = time required to walk distance, and $1\frac{1}{2}\div x$ = time for $1\frac{1}{2}$ miles;

$\therefore\ 1\frac{1}{2}\div x+\frac{1}{3}+3\div(x+1\frac{1}{2})=4\frac{1}{2}\div x$, whence $x=3$.

10. (1) $\therefore\ \dfrac{a^4}{b^4}=\dfrac{c^4}{d^4}=\dfrac{a^4+c^4}{b^4+d^4}$; $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2}\ \therefore\ \dfrac{a^2c^2}{b^2d^2}=\dfrac{c^4}{d^4}$

$\therefore\ (a^4+c^4)\div(b^4+d^4)=a^2c^2\div b^2d^2$. (2) 2000.

(3) Quantity vanishes for $a+b=0$, $\therefore\ (a+b)(b+c)(c+a$. Numerical factor $=3$.

ADDENDA.

NOTE.—*References are to the "Hand Book."*

Q. 5, page 14. By No. 2, Ex. V. $(ax+by+cz)^2+(ay-bx)^2$
$=(a^2+b^2)(x^2+y^2)+c^2z^2+2cz(ax+by)$. Also
$(cx-az)^2+(bz-cy)^2=c^2(x^2+y^2)+z^2(a^2+b^2)-2cz(ax+by)$.
Now add, $\therefore$ &c.

9, p. 14. By Ex. I., p. 11. $2(a^2+b^2-x^2)^2+(a_1^2+b_1^2-x^2)^2$
$=(a^2+b^2+a_1^2+b_1^2-2x^2)^2+\{(a^2+b^2)-(a_1^2+b_1^2)\}^2$. Also
$\{(a^2+b^2)-(a_1^2+b_1^2)\}^2-4(ab+a_1b_1)^2=\{(a+b)^2-(a_1-b_1)^2\times$
$\{(a-b)^2-(a_1+b_1)^2\}=(a+b+a_1-b_1)(a+b-a_1+b_1)\times$
$(a-b+a_1+b_1)(a-b-a_1-b_1)$;
$\therefore\ 2\{(a^2+b^2-x^2)^2+(a_1^2+b_1^2-x^2)^2-2(ab+a_1b_1)^2\}$
$=(a^2+b^2+a_1^2b_1^2-2x^2)^2-(a+b+a_1-b_1)$
$\times(a+b-a_1+b_1)(-a+b+a_1+b_1)$;
similarly for $2\{(a^2+a_1^2-x^2)^2+\&c.\}$.

5, p. 43, is not a case of exact division: remainder can be easily found.

70, p. 164. $$\frac{a+b}{x-b}-\frac{b+d}{x-(a+b+2c+d)}=$$

$$\frac{a+c}{x-c}-\frac{c+d}{x-(a+2b+c+d)}\ \therefore\ \frac{(a-d)x-a(a+2b+2c+d)-2bc}{(x-b)\{x-(a+b+2c+d)\}}$$

$$=\frac{(a-d)x-a(a+2b+2c+d)-2bc}{(x-c)\{x-(a+2b+c+d)\}}$$

$\therefore\ (a-d)x-a(a+2b+2c+d)-2bc=0$, unless the denominators are equal, *i.e.*, $b(a+b+2c+d)=c(a+2b+c+d)$, which reduces to $b=c$, or $a+b+c+d=0$.

80, p. 165. $\therefore\ m(a-b)\left\{\frac{m}{x-m}-\frac{q}{x-q}\right\}+$

$$n(b-c)\left\{\frac{n}{x-n}-\frac{q}{x-q}\right\}+p(c-d)\left\{\frac{p}{x-p}-\frac{q}{x-q}\right\}=0$$

$$\therefore\ \frac{m(a-b)(m-q)x}{(x-m)(x-q)}+\frac{n(b-c)(n-q)x}{(x-n)(x-q)}+\frac{p(c-d)(p-q)x}{(x-p)(x-q)}=0.$$

$\therefore\ x=0$ {or, $m(a-b)(m-g)\div(x-m)(x-q)+\&c.=0$}.

Pp. 175, 176, 177, Exercise LVI.

75–87. Let $a-x=k+y$, $x-b=k-y$, $a=k+m$ and $\therefore$ $-b=k-m$, from which it follows that $2k=a-b$, $2m=a+b$, and $y=\frac{1}{2}(a+b)-x=m-x$. Also let j be defined by the equation $j^2+1=0$.

77. $\dfrac{(k+y)^4+(k-y)^4}{(k+y)^2+(k-y)^2}=\dfrac{(k+m)^4+(k-m)^4}{(k+m)^2+(k-m)^4}$

$$\therefore\ \frac{k^4+6k^2y^2+y^4}{k^4+6k^2m^2+m^4}=\frac{k^2+y^2}{k^2+m^2}$$

$$\therefore\ \frac{y^4-m^4+6k^2(y^2-m^2)}{k^4+6k^2m^2+m^4}=\frac{y^2-m^2}{k^2+m^2}$$

$\therefore\ y^2-m^2=0$, and $\therefore\ y_1+m=0$, or $y_1-m=0$. (By A)

or $\dfrac{y^2+m^2+6k^2}{k^4+6k^2m^2+m^4}=\dfrac{1}{k^2+m^2}$, and $\therefore$

$$y^2+\frac{(5k^2+m^2)k^2}{k^2+m^2}=0,\ \text{or}\ y^2-\frac{j^2(5k^2+m^2)k^2}{k^2+m^2}=0,$$

$\therefore\ y_3=jkr$, $y_4=-jkr$, in which $r^2=(5k^2+m^2)\div(k^2+m^2)$.

78. $\therefore\ \dfrac{(k+y)^4+(k-y)^4}{(k+y)^3+(k-y)^3}=\dfrac{(k+m)^4+(k-m)^4}{(k+m)^3+(k-m)^3}$

$$\therefore\ \frac{k^4+6k^2y^2+y^4}{k^4+6k^2m^2+m^4}=\frac{k^3+3ky^2}{k^3+3km^2}$$

$$\therefore\ \frac{y^4-m^4+6k^2(y^2-m^2)}{k^4+6k^2m^2+m^4}=\frac{3k(y^2-m^2)}{k^3+3km^2}$$

$\therefore\ y^2-m=0$, and $\therefore\ y_1+m=0$, $y_2-m=0$,

or $\dfrac{y^2+m^2+6k^2}{k^4+6k^2m^2+m^4}=\dfrac{3}{k^2+3m^2}$ or $y^2+\dfrac{k^2(3k^2+m^2)}{k^2+3m^2}=0.$

79. $\therefore \dfrac{(k+y)^5+(k-y)^5}{(k+y)^3+(k-y)^3} = \dfrac{(k+m)^5+(k-m)^5}{(k+m)^3+(k-m)^3}$

$$\therefore \frac{k^5+10k^3y^2+5ky^4}{k^5+10k^3m^2+5km^4} = \frac{k^3+3ky^2}{k^3+3km^2}$$

$$\therefore \frac{5(y^4-m^4)+10k^2(y^2-m^2)}{k^4+10k^2m^2+5m^4} = \frac{3(y^2-m^2)}{k^2+3m^2}$$

$\therefore y^2-m^2=0$, and $\therefore y_1+m=0,\ y_2-m=0,$

or $\dfrac{5(y^2+m^2)+10k^2}{k^4+10k^2m^2+5m^4} = \dfrac{3}{k^2+3m^2}.$

80. $\therefore \dfrac{(k+y)^4+(k-y)^4}{y^2-k^2} = \dfrac{(k+m)^4+(k-m)^4}{m^2-k^2}$

$$\therefore \frac{k^4+6k^2y^2+y^4}{k^4+6k^2m^2+m^4} = \frac{y^2-k^2}{m^2-k^2}$$

$$\therefore \frac{y^4-m^4+6k^2(y^2-m^2)}{k^4+6k^2m^2+m^4} = \frac{y^2-m^2}{m^2-k^2}$$

$\therefore y^2-m^2=0$, or $\dfrac{y^2+m^2+6k^2}{k^4+6k^2m^2+m^4} = \dfrac{1}{m^2-k^2}.$

81. $\therefore \dfrac{(k+y)^3+(k-y)^3}{(k^2-y^2)^2} = \dfrac{(k+m)^3+(k-m)^3}{(k^2-m^2)}$

$$\therefore \frac{k^2+3y^2}{k^2+3m^2} = \frac{k^4-2k^2y^2+y^4}{k^4-2k^2m^2+m^4}$$

$$\therefore \frac{3(y^2-m^2)}{k^2+3m^2} = \frac{k^4-m^4-2k^2(y^2-m^2)}{k^4-2k^2m^2+m^4}$$

$\therefore y^2-m^2=0$, or $\dfrac{3}{k^2+3m^2} = \dfrac{y^2+m^2-2k^2}{k^4-2k^2m^2+m^4}.$

82. $\therefore \dfrac{(h+y)^4+(k-y)^4}{4y^2} = \dfrac{(k+m)^4+(k-m)^4}{4m^2}$

$\therefore \dfrac{k^4+6k^2y^2+y^4}{k^4+6k^2m^2+m^4} = \dfrac{y^2}{m^2} \quad \therefore \dfrac{y^4-m^4+6k^2(y^2-m^2)}{k^4+6k^2m^2+m^4} = \dfrac{y^2-m^2}{m^2},$

$\therefore y^2-m=0$, or $\dfrac{y^2+m^2+6k^2}{k^4+6k^2m^2+m^4} = \dfrac{1}{m^2}$ *i.e.*, $m^2y^2-k^4=0,$

$\therefore y_1+m=0,\ y_2-m=0,\ my_3+k^2=0,\ my_4-k^2=0.$

83. $\therefore \dfrac{(k+y)^5+(k-y)^5}{4y^2} = \dfrac{(k+m)^5+(k-m)^5}{4m^2}$,

$\therefore \dfrac{k^4+10k^2y^2+5y^4}{k^4+10k^2m^2+5m^4} = \dfrac{y^2}{m^2}$,

$\therefore \dfrac{5(y^4-m^4)+10k^2(y^2-m^2)}{k^4+10k^2m^2+5m^4} = \dfrac{y^2-m^2}{m^2}$ $\therefore y^2-m^2=0$,

or $\dfrac{5(y^2+m^2)+10k^2}{k^4+10k^2m^2+5m^4} = \dfrac{1}{m^2}$, *i.e.*, $5m^2y^2-k^4=0$,

$\therefore y_1+m=0,\ y_2-m=0,\ 5my_3+k^2\sqrt{5}=0,\ 5my_4-k^2\sqrt{5}=0.$

84. $\therefore \dfrac{(k+y)^5+(k-y)^5}{(k+y)^2+(k-y)^2} = 8k^3$,

$\therefore 5y^4+10k^2y^2+k^4=8k^2(y^2+k^2)$,

$\therefore 5y^4+2k^2y^2-7k^4=0, \quad \therefore (y^2-k^2)(5y^2+7k^2)=0$,

$\therefore (y+k)(y-k)(5y+jk\sqrt{35})(5y-jk\sqrt{35})=0.$

85. $\therefore \dfrac{(k+y)^4-(k-y)^4}{(k+y)-(k-y)} = \dfrac{2kc}{k^2-y^2}$

$\therefore \dfrac{4ky(k^2+y^2)}{2y} = \dfrac{2kc}{k^2-y^2}$

$\therefore k^4-y^4=c$, or $y^4-(k^4-c)=0$,

$\therefore \{y^2-\sqrt{(k^4-c)}\}\{y^2-j^2\sqrt{(k^4-c)}\}=0$,

$\therefore \{y+\sqrt[4]{(k^4-c)}\}\{y-\sqrt[4]{(k^4-c)}\}\{y+j\sqrt[4]{(k^4-c)}\}\times$
$\{y-j\sqrt[4]{(k^4-c)}\}=0.$

86. $\dfrac{(k+y)^5+(k-y)^5}{(k+y)^2+(k-y)^2}=c(k^2-y^2)$,

$\therefore k^5+10k^3y^2+5ky^4=c(k^4-y^4)$,

$\therefore (5k+c)y^4+10k^3y^2+(k-c)k^4=0.$ Let $ks^2=5k+c$,

$\therefore s^4y^4+10k^2s^2y^2+(6-s^2)s^2k^4=0$,

$\therefore (s^2y^2+5k^2)^2-\{25-6s^2+s^4\}k^4=0.$

Let $r^2=25-6s^2+s^4$, $\therefore \{s^2y^2-(r-5)k^2\}\{s^2y^2+(r+5)k^2\}=0$

$\therefore \{sy-k\sqrt{(r-5)}\}\{sy+k\sqrt{(r-5)}\}\{sy-jk\sqrt{(r+5)}\}\times$
$\{sy+jk\sqrt{(r+5)}\}=0.$

87. $\dfrac{(k+y)^3+(k-y)^3}{(k+y)^4+(k-y)^4} = \dfrac{c}{k^2-y^2}$,

$\therefore k(k^2+3y^2)(k^2-y^2)=c(k^4+6k^2y^2+y^4)$

$\therefore (3k+c)y^4-2k^2(k-3c)y^2-k^4(k-c)=0.$

Let $s^2=3k+c$, $\therefore s^4y^4-2k^2(k-3c)s^2y^2-k^4(k-c)s^2=0.$

Let $r^2=(k-3c)^2+(k-c)(3k+c)$,

$\therefore \{s^2y^2-k^2(k-3c+r)\}\{s^2y^2-k^2(k-3c-r)\}=0,$

$\therefore sy\pm k\sqrt{(k-3c\pm r)}=0.$

88. Expand and collect, $3x^2(x^2+2x+1)-8=0.$

$\therefore 9x^2(x+1)^2-24=0$, $\therefore 3x(x+1)-\sqrt{24}=0$, (By B.)

or $3x(x+1)+\sqrt{24}=0$, $\therefore 6x+3\pm\sqrt{(9\pm12\sqrt{24})}=0.$ (By D.)

89–102. Work with a new variable w such that $wx=x^2+1$. Having determined the value of w, that of x may be determined

thus $\dfrac{x^2+1}{x}=w$, $\therefore \left(\dfrac{x+1}{x-1}\right)^2=\dfrac{w+2}{w-2}=r^2$ say

$\therefore \dfrac{x+1}{x-1}=\pm r$ $\therefore x_1=\dfrac{r+1}{r-1}$, $x_2=\dfrac{r-1}{r+1}$.

89. $\therefore \dfrac{(x^2+1)^2-2x^2}{2x(x^2+1)}=\dfrac{a}{b}$ $\therefore \dfrac{w^2-2}{2w}=\dfrac{a}{b}$ (α)

$\therefore \dfrac{w^4-4w^2+4}{4w^2}=\dfrac{a^2}{b^2}$ $\therefore \dfrac{w^4+4w^2+4}{4w^2}=\dfrac{a^2+2b^2}{b^2}=\dfrac{s}{b^2}$ say,

$\therefore \dfrac{w^2+2}{2w}=\dfrac{\pm s}{b}$ (β) $\therefore (\alpha)+(\beta)$, $w=\dfrac{a\pm s}{b}$

$\therefore \dfrac{w+2}{w-2}=\dfrac{a+2b\pm s}{a-2b\pm s}=r^2$, (as above).

90. $\therefore \dfrac{(w+2)w}{(w-2)(w-1)}=\dfrac{a}{b}$ or $\dfrac{w^2+2w}{w^2-3w+2}=\dfrac{a}{b}$

$\therefore \dfrac{w^2+2w}{5w-2}=\dfrac{a}{a-b}$. Let $t=5w-2$, $\therefore 5w=t+2.$

$\therefore \dfrac{t^2+14t+24}{t}=\dfrac{25a}{a-b}$ $\therefore \dfrac{t^2+24}{t}=\dfrac{11a+14b}{a-b}$ (α)

$$\therefore \frac{t^2-24}{t} = \frac{\pm\sqrt{\{(11a+14b)^2-96(a-b)^2\}}}{a-b} =$$

$$\frac{\pm 5\sqrt{(a^2+20ab+4b^2)}}{a-b} = \frac{\pm 5s}{a-b} \text{ say.} \quad (\beta)$$

$$\therefore (\alpha)+(\beta), \quad t = \frac{11a+14b\pm 5s}{2(a-b)} = 5w-2$$

$$\therefore w = \frac{3a+2b\pm s}{2(a-b)}.$$

91. $\dfrac{w-2}{(w-1)^2} = \dfrac{a}{b} \quad \therefore \dfrac{w^2-6w+9}{(w-1)^2} = \dfrac{b-4a}{b} = s^2$ say,

$$\therefore \frac{w-3}{w-1} = \pm s \quad \therefore w = \frac{3\pm s}{1\pm s}.$$

92 $\therefore \dfrac{(w+1)^2}{w(w+2)} = \dfrac{a}{b} \quad \therefore (w+1)^2 = \dfrac{a}{a-b}.$

93. $\dfrac{w^2}{w^2+2} = \dfrac{a}{b} \quad \therefore w^2 = \dfrac{2a}{b-a}.$

94. $\dfrac{(w+2)^2}{w} = \dfrac{a}{b} \quad \therefore \left(\dfrac{w+2}{w-2}\right)^2 = \dfrac{a}{a-8b} = s^4$, say

$$\therefore \left(\frac{x+1}{x-1}\right) = s^4, \qquad \therefore \frac{x+1}{x-1} = \pm s \text{ or } \pm js.$$

95. $\therefore \dfrac{w+2}{(w-2)^2} = \dfrac{a}{b}.$ Let $t=w-2$.

$$\therefore \frac{t+4}{t^2} = \frac{a}{b} \quad \therefore \frac{t^2+16t+64}{t^2} = \frac{16a+b}{b} = s^2 \text{ say,}$$

$$\therefore \frac{t+8}{t} = \pm s, \therefore \frac{t+4}{t} = \frac{1\pm s}{2} = \frac{w+2}{w-2}.$$

96. $\dfrac{w+1}{w-2} \cdot \dfrac{w-1}{w-2} = \dfrac{w^2-1}{w^2-4} = \dfrac{a}{b} \therefore w^2 = \dfrac{4a-b}{a-b} = \dfrac{s^2}{(a-b)^2}$ say,

$$\therefore w = \frac{\pm s}{a-b} \qquad \therefore \frac{w+2}{w-2} = \frac{2(a-b)\pm s}{2(b-a)\pm s}.$$

97. $\dfrac{w^2-3}{w^2-4} = \dfrac{a}{b}, \therefore w^2 = \dfrac{4a-3b}{a-b}.$

98. $\dfrac{w^2-4}{w} = \dfrac{b}{a} \therefore \left(\dfrac{w^2+4}{w}\right)^2 = \dfrac{b^2+16a^2}{a^2} = \dfrac{s^2}{a^2}$ say

$\therefore\ w = \dfrac{b \pm s}{2a}.$

99. $\dfrac{(w+2)(w-1)}{(w-2)(w+1)} = \dfrac{a}{b}$, or $\dfrac{w^2+w-2}{w^2-w-2} = \dfrac{a}{b}$,

$\therefore\ \dfrac{w^2-2}{w} = \dfrac{a+b}{a-b} \therefore \left(\dfrac{w^2+2}{w}\right)^2 = \dfrac{(a+b)^2+8(a-b)}{(a-b)^2} = \dfrac{s^2}{(a-b)^2}$ say,

$\therefore\ w = \dfrac{a+b \pm s}{2(a-b)}.$

100. $\dfrac{x^4+x^3+x^2+x+1}{x^4-x^3+x^2-x+1} = \dfrac{(x^2+1)^2+x(x^2+1)-x^2}{(x^2+1)^2-x(x^2+1)-x^2}$

$\therefore\ \dfrac{w^2+w-1}{w^2-w-1} = \dfrac{a}{b} \therefore \dfrac{w^2-1}{w} = \dfrac{a+b}{a-b},$

$\therefore\ \left(\dfrac{w^2+1}{w}\right)^2 = \dfrac{(a+b)^2+4(a-b)^2}{(a-b)^2} = \dfrac{s^2}{(a-b)^3}$ say,

$\therefore\ w = \dfrac{a+b \pm s}{2(a-b)}.$

101. $\dfrac{(w+2)^2}{w^2-2} = \dfrac{a}{b} \therefore \dfrac{(w+2)^2}{w^2+2w+1} = \dfrac{2a}{a+b} = s^2$ say,

$\therefore\ \dfrac{w+2}{w+1} = \pm s, \therefore \dfrac{w+2}{w-2} = \dfrac{\pm s}{4 \pm 3s}.$

102. $\dfrac{(w+2)^2}{w^2-w-1} = \dfrac{a}{b} \therefore \dfrac{(w+2)^2}{w^2} = \dfrac{5a}{a+4b} = s^2$ say,

$\therefore\ \dfrac{w+2}{w} = \pm s, \therefore \dfrac{w+2}{w-2} = \dfrac{\pm s}{2 \mp s}.$

103. $(a-x)(x-b)w = (a-x)^2+(x-b)^2$

$2\{(a-x)^4+(x-b)^4\} - 9(a-x)(x-b)\{(a-x)^2+(x-b)^2\} +$
$14(a-x)^2(x-b)^2 = 0, \therefore 2(w^2-2)-9w+14 = 0,$

or $2w^2-9w+10=0$. or $(w-2)(2w-5)=0$,

$\therefore w_1 = 2,\ 2w^2 = 5,\ \therefore 2(a-x)(x-b) = (a-x)^2+(x-b)^2,$

and $\therefore \{(a-x)-(x-b)\}^2=0, \therefore x_1=\frac{1}{2}(a+b)$,

or $5(a-x)(x-b)=2\{(a-x)^2+(x-b)^2\}$,

and $\therefore \{2(a-x)-(x-b)\}\{(a-x)-2(x-b)\}=0$,

$\therefore x_2=\frac{1}{3}(2a+b), x_3=\frac{1}{3}(a+2b)$.

104. $4\{(a-x)^4-2(a-x)^2(x-b)^2+(x-b)^4\}-$
$9(a-x)^2(x-b)^2=0$,

$\therefore [2\{(a-x)^2-(x-b)^2\}-3(a-x)(x-b)\}\times$
$[2\{(a-x)^2-(x-b)^2\}+3(a-x)(x-b)]=0$,

$\therefore \{2(a-x)+(x-b)\}\{(a-x)-2(x-b)\}\times$
$\{2(a-x)-(x-b)\}\{(a-x)+2(x-b)\}=0$,

$\therefore x_1=2a-b, x_2=\frac{1}{3}(a+2b), x_3=\frac{1}{2}(2a+b), x_4=2b-a$.

105. $x^4-12x^3+49x^2-78x+40=0$,

$\therefore (x^2-6x)^2+13(x^2-6x)+40=0$,

$(x^2-6x+5)(x^2-6x+8)=0$,

$\therefore (x-1)(x-5)(x-2)(x-4)=0, \therefore x=1$ or 2, or 4, or 5.

106. $(x^4-1)-6x(x^2-1)+7(x^2-1)=0$,

$\therefore (x^2-1)\{x^2+1-6x+7)=0$,

$\therefore (x-1)(x+1)(x-2)(x-4)=0, \therefore x=+1, -1, 2$, or 4.

107. $(x^2-5x)^2+10(x^2-5x)+24=0$,

$\therefore (x^2-5x+4)(x^2-5x+6)=0$.

$\therefore (x-1)(x-4)(x-2)(x-3)=0, \therefore x=1, 2, 3$, or 4.

108. $2(4x^2-3x)^2-7(4x^2-3x)+5=0$,

$\therefore (8x^2-6x-5)(4x^2-3x-1)=0$,

$\therefore (2x+1)(4x-5)(4x+1)(x-1)=0, \therefore x=-\frac{1}{2}, -\frac{1}{4}, 1$ or $\frac{5}{4}$.

109. $(x^3+1)-6(x^2-1)+5(x+1)=0$,

$\therefore (x+1)\{(x^2-x+1)-6(x-1)+5\}=0$,

$\therefore (x+1)(x^2-7x+12)=0$, or $(x+1)(x-3)(x-4)=0$,

$\therefore x=-1, 3$, or 4.

110. $\dfrac{5(x+x-4a)}{x^2-4ax} - \dfrac{4(x-a+x-3a)}{x^2-4ax+3a^2} - \dfrac{9(x-2a)}{x^2-4ax+4a^2} = 0,$

Let $x^2-4ax=ay^2$ $\therefore$ $\dfrac{10}{y} - \dfrac{8}{y+3} - \dfrac{9}{y+4} = 0,$

$\therefore$ $7y^2-11y-120=0$, or $(y-5)(7y+24)=0$,

$\therefore$ $x^2-4ax=5a^2$, or $7x^2-28ax+24a^2=0$,

$\therefore$ $(x-5a)(x+a)=0$,

$\therefore$ $x_1=5a$, $x_2=-a$. (Two other values.)

111. Let $25y=x^2-35$.

$\dfrac{14(20+55)}{25y-1100} + \dfrac{5(5+40)}{25y-200} - \dfrac{4(25-10)}{25y+250} = 0.$

$\therefore$ $\dfrac{70}{y-44} + \dfrac{15}{y-80} - \dfrac{4}{y+10} = 0,$

$\therefore$ $y^2-2y-168=0$, or $(y+12)(y-14)=0$,

$\therefore$ $x^2-35x+300=0$, or $x^2-35x-350=0$,

$\therefore$ $(x-15)(x-20)=0$,

$\therefore$ $x=15$ or 20. Two other values.

112. $\therefore$ $\dfrac{5a}{x} - \dfrac{9a}{x-a} + \dfrac{2a}{x-2a} = \dfrac{2a}{x-3a} - \dfrac{9a}{x-4a} + \dfrac{5a}{x-5a}.$

Let $a^2y=x^2-5ax$, $\therefore$ $\dfrac{25}{y} - \dfrac{27}{y+4} + \dfrac{2}{y+6} = 0.$

$\therefore$ $4y=25$, $\therefore$ $4x^2-20ax+25a^2=0$,

$\therefore$ $(2x-5a)^2=0$, $\therefore$ $x=2\frac{1}{4}a$.

113. $\dfrac{2}{x+2} + \dfrac{2}{x} + \dfrac{5}{x-1} = \dfrac{5}{x-2} + \dfrac{2}{x-3} + \dfrac{2}{x-5}.$

Let $y=x^2-3x$ and collecting pairs of terms as in preceding solutions, $\dfrac{14}{y-10} + \dfrac{6}{y} + \dfrac{6}{y+2} = 0,$

$\therefore$ $5y^2-14y-24=0$, $\therefore$ $(y-4)(5y+6)=0$,

$\therefore$ $x^2-3x-4=0$, or $5x^2-15x+6=0$,

$\therefore$ $(x-4)(x+1)=0$, $\therefore$ $x=4$ or -1.

114. Assume $y=x^2-6x$, and collecting in pairs and reducing $\left(\text{the term } \dfrac{8}{x-3} \text{ must be written } \dfrac{8(x-3)}{x^2-6x+9}\right)$ the equation becomes $\dfrac{7}{y} - \dfrac{31}{y+5} + \dfrac{20}{y+8} + \dfrac{4}{y+9} = 0,$

$$\therefore\ 41y^2+73y-2520=0 \text{ or } (y-7)(41y+360)=0,$$

$$\therefore\ x^2-6x-7=0 \text{ or } 41x^2-246x+360=0,$$

$$\therefore\ (x-7)(x+1)=0, \quad \therefore\ x=7 \text{ or } -1.$$

115. $\therefore\ \sqrt{(x^2-a^2-b^2)}+\sqrt{(x^2-b^2-c^2)}=x-\sqrt{(x^2-c^2-a^2)}$

$$\therefore\ (x^2-a^2-b^2)+(x^2-b^2-c^2)+2\sqrt{\{x^4-(a^2+2b^2+c^2)x^2+(a^2+b^2)(b^2+c^2)\}}=x^2+(x^2-c^2-a^2)-2x\sqrt{(x^2-c^2-a^2)},$$

$$\therefore\ b^2-\sqrt{\{x^4-\&c.\}}=x\sqrt{(x^2-c^2-a^2)}$$

$$\therefore\ b^4+x^4-(a^2+2b^2+c^2)x^2+(a^2+b^2)(b^2+c^2)-2b^2\sqrt{\{x^4-\&c\}} =x^4-(c^2+a^2)x^2$$

$$\therefore\ 2b^2x^2-\{a^2b^2+b^2c^2+c^2a^2+2b^4\}=-2b^2\sqrt{\{x^4-\&c.\}}$$

$$\therefore\ 4b^4x^4-4b^2x^2(a^2b^2+b^2c^2+c^2a^2+2b^4)+(a^2b^2+b^2c^2+c^2a^2+2b^4)^2 =$$
$$4b^4x^4-4b^4x^2(a^2+2b^2+c^2)+4b^4(a^2b^2+b^2c^2+c^2a^2+b^4)$$

$$\therefore\ 4a^2b^2c^2x \quad (a^2b^2+b^2c^2+c^2a^2)^2=0,$$

$$\therefore\ x=\pm\tfrac{1}{2}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right).$$

116. Multiply both numerator and denominator of each member by the denominator of that member and reduce

$$\therefore\ \frac{a^2}{a^2-\sqrt{(a^4-4x^2)}}=\frac{m^2x}{m^2x-\sqrt{(m^4x^2-4)}}$$

$$\therefore\ \frac{a^2}{\sqrt{(a^4-4x^2)}}=\frac{m^2x}{\sqrt{(m^4x^2-4)}}. \quad \text{Square and reduce}$$

$$m^4x^4=a^4 \qquad \therefore\ mx=\pm a \text{ or } \pm ja.$$

117. Work as in Ex. 115.

Let $2s=a+b+c$ and $2s'^2=a^2+b^2+c^2$.

$$x=\frac{2s(s-a)(s-b)(s-c)}{\sqrt{\{(s'^2-a^2)(s'^2-b^2)(s'^2-c^2)\}}}.$$

Or thus, let $a_1^2 = \frac{1}{2}(-a^2+b^2+c^2)$, $b_1^2=\frac{1}{2}(a^2-b^2+c^2)$, $c_1^2 = \frac{1}{2}(a^2+b^2-c^2)$, and the equation becomes

$$\sqrt{(x^2-a_1^2-b_1^2)}+\sqrt{(x^2-b_1^2-c_1^2)}+\sqrt{(x^2-c_1^2-a_1^2)}=x,$$

of which (see Ex. 115 above) the solution is

$$x=\pm\tfrac{1}{2}\left(\frac{b_1c_1}{a_1}+\frac{c_1a_1}{b_1}+\frac{a_1b_1}{c_1}\right).$$

118. $\therefore\ \sqrt{(a-x)}-\sqrt{(b-x)}=c,$

$\therefore\ a-x=c^2+b-x+2c\sqrt{(b-x)},$

$\therefore\ 2c\sqrt{(b-x)}=a-b-c^2,$

$$\therefore\ x=\frac{2ab+2ac^2+2bc^2-a^2-b^2-c^4}{4c^2}.$$

119. $\therefore\ \dfrac{\sqrt[3]{(a-x)}}{(\sqrt[3]{x-b})}=\dfrac{a-x}{x-b}\quad\therefore\quad\dfrac{a-x}{x-b}=\dfrac{(a-x)^3}{(x-b)^3},$

$\therefore\ (a-x)(x-b)\{(a-x)^2-(x-b)^2\}=0,$

$\therefore\ (a-x)(x-b)(a-b)(a+b-2x)=0,$

$\therefore\ x=a$ or b or $\frac{1}{2}(a+b)$.

120 $(w+v)^5=w^5+v^5+5wv(w+v)^3-5w^2v^2(w+v).$

Let $w=\sqrt[5]{(a+x)}$ and $v=\sqrt[5]{(a-x)}$, $\therefore\ w+x=\sqrt[5]{(2a)}$,

$\therefore\ 2a=2a+5\sqrt[5]{(a^2-x^2)}.\sqrt[5]{(8a^3)}-5\sqrt[5]{(w^2-x^2)^2}\sqrt[5]{(2a)},$

$\therefore\ \sqrt[5]{\{4a^2(a^2-x^2)\}}=\sqrt[5]{(a^2-x^2)^2},$

$\therefore\ 4a^2(a^2-x^2)=(a^2-x^2)^2,\ \therefore\ (a^2-x^2)(3a^2+x^2)=0,$

$\therefore\ x=\pm a$ or $\pm ja\sqrt{3}$.

121. $\therefore\ \dfrac{(w^2+v^2)^2}{w+v}=w^3+v^3,$

$\therefore\ w^4+2w^2v^2+v^4=w^4+wv(w^2+v^2)+v^4,$

$\therefore\ wv(w-v)^2=0,$

$\therefore\ w=0$ or $v=0$, or $w=v$. Cube these.

$\therefore\ a-x=0$ or $x-b=0$, or $a-x=x-b$,

$\therefore\ x=a$ or b or $\frac{1}{2}(a+b)$.

21. p. 200, $\therefore (x+y)+(y+z)+(z+z)=2(a+b+c)$, and

$$c(x+y)+a(y+z)+b(z+x)=2(a^2+b^2+c^2),$$

$$b(x+y)+c(y+z)+a(z+x)=2(ab+bc+ca), \text{ or}$$

$$\begin{aligned} &u+v+w=a+b+c && \therefore\ u=c,\\ &bu+cv+aw=ab+bc+ca && v=a,\\ &cu+av+bw=a^2+b^2+c^2 && w=b, \text{ \&c.}\end{aligned}$$

15. p. 205,

$$\therefore \frac{x^2+x+1}{y^2+y+1}=\frac{a^2+a+1}{b^2+b+1} \quad \therefore \quad \frac{(x-a)(x+a+1)}{(y-b)(y+b+1)}=\frac{a^2+a+1}{b^2+b+1} \ldots (3).$$

Assume $x-1=(a-1)v$ and $\therefore\ y-1=(b-1)v$, $\therefore$ also

$$x-a=(a-1)(v-1) \text{ and } x+a+1=(a-1)(v+1)+3$$

$$y-b=(b-1)(v-1) \text{ and } y+b+1=(b-1)(v+1)+3.$$

$$(3)\ \therefore \frac{(a-1)(v-1)\{(a-1)(v+1)+3\}}{(b-1)(v-1)\{(b-1)(v+1)+3\}}=\frac{a^2+a+1}{b^2+b+1}$$

$\therefore\ v-1=0$, and $\therefore\ x=a$, $y=b$; or

$$\frac{(a-1)^2(v+1)+3(a-1)}{(b-1)^2(v+1)+3(b-1)}=\frac{a^2+a+1}{b^2+b+1}$$

$$\therefore \frac{3a(v+1)+3(a-1)}{3b(v+1)-3(b-1)}=\frac{a^2+a+1}{b^2+b+1}$$

$$\text{or } \frac{av+1}{bv+1}=\frac{a^2+a+1}{b^2+b+1} \quad \therefore\ v=\frac{a+b+1}{1-ab} \quad \therefore\ x$$

$$=(a^2-b)\div(1-ab),\ y=(b^2-a)\div(1-ab).$$

Errata.

Example iii., question 28, "Cube of the sum."